



## SYNCHRONIZATION OF ECONOMIC SYSTEMS WITH FRACTIONAL ORDER DYNAMICS USING ACTIVE SLIDING MODE CONTROL

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### ABSTRACT

*Synchronization of chaos has widely spread as an important issue in nonlinear systems and is one of the most important branches on the problem of controlling of chaos. In this paper, among different chaotic systems the economy chaotic system has been selected. The main aim of this paper is the designing based on the active sliding mode control for the synchronization of fractional-order chaotic systems. The chaos in the economic series could have serious and very different consequences in common macro-economy models. In this paper, this article expressed the various positions of synchronization in economic system that include of changes in the coefficients of the system, changes in the initial conditions of the system and different fractional-order synchronization on the economic system in which Synchronization is shown in some examples.*

**Keywords:** Economic system, Chaos, Chaos control, Synchronization, Active sliding mode controller, Fractional order.

### 1. INTRODUCTION

Synchronization between two systems is one of the important processes in the control of complex phenomena for chemical, physical, and biological systems. The goal of complete synchronization or anti-synchronization is to synchronize the states of slave system identical or opposite to the states of master system, many approaches were reported to study chaos synchronizations for certain types of chaotic attractors such as active control (Chen *et al.*, 2009), adaptive control (Li *et al.*, 2010; Yang, 2012). In the last few years, economy physics has been raised to an alternative scientific methodology to understand the highly complex dynamics of real financial and economic systems. Researchers are striving to explain the central features of economic data: irregular microeconomic fluctuations, erratic macroeconomic fluctuations (business cycles), irregular growth, structural changes, and overlapping waves of economic development. In

recent years, the importance of chaos in economics has tremendously increased: “chaos represents a radical change of perspective on business cycles” (Serletic, 1996). Chaos is the inherent randomness in a definite system. The randomness is caused by system internals, not by external disturbances. The main features of deterministic chaos, such as the complex patterns of phase portraits and positive Lyapunov exponents, have been found in many economic aggregate data such as the gross national product (Chen, 1988). Many continuous chaotic models have been proposed to study complex economic dynamics in the literature, e.g., the forced van der Pol model (Chian, 2000; Chian *et al.*, 2006), the IS-ML model (Fanti and Manfredi, 2006). In the last few years, economy physics has been raised to an alternative scientific methodology to understand the highly complex dynamics of real financial and economic systems. Meanwhile, most of previous studies have been shown that some fractional-order systems exhibit chaotic behavior (Ahmad and Sprott, 2003; Zhou and Ding, 2012). Some approaches based on this configuration have been attained to achieve chaos synchronization in fractional-order chaotic systems, such as PC control (Li and Deng, 2006). Active control (Bhalekar and Daftardar-Gejji, 2010). sliding mode control (SMC) (Chen *et al.*, 2012), etc. In which, the sliding mode controller has some attractive advantages, including fast dynamic responses and good transient performance; external disturbance rejection; insensitivity to parameter variations and model uncertainties (Slotine and Li, 1991; Boiko *et al.*, 2006). In addition, SMC method plays an important role in the application to practical problems. For example, in (Tavazoei and Haeri, 2008), Tavazoei MS and his co-operator proposed a controller based on active sliding mode theory to synchronize chaotic fractional-order systems in master-slave structure. In (Wang *et al.*, 2012), the problem of modified projective synchronization of fractional-order chaotic system was considered, and finite-time synchronization of non-autonomous fractional-order uncertain chaotic systems was investigated by Aghababa MP in (Aghababa, 2012). Also, that all fluctuations in financial variables are correlated with all future fluctuations. This was our motivation for describing financial systems using a fractional nonlinear model since it simultaneously possesses memory and chaos. Chaotic attractors have been found in fractional-order systems (Ge and Zhang; Hartley *et al.*, 1995) in the past decade. Recently, the present author and Chen (Chen and Chen) investigated the chaotic behavior of the van der Pol equation with physically fractional damping and study examines the two most attractive characteristics, memory and chaos, in simulations of financial systems (Wei-Ching, 2008). In this paper, among different chaotic systems the economy chaotic system has been selected. The main aim of this paper is the designing based on the active sliding mode control for the synchronization of fractional-order chaotic systems. This article expressed the various positions of synchronization in economic system that include of changes in the coefficients of the system, changes in the initial conditions of the system and different fractional-order synchronization on the economic system. In which Synchronization is shown in some examples.

## 2. CHAOTIC SYSTEMS

Chaos is a long-term behavior of no periodic in a Deterministic system which shows sensitive dependence on initial conditions. By long-term non-periodic behavior of the dynamical systems we mean that there are paths when time tends to infinity, the paths of these systems are not leading to fixed points, periodic orbits and quasi-periodic orbits. By being sensitive to the initial conditions in dynamical systems, we mean that the adjacent channels are separated rapidly and dramatically. In fact, this feature is the main difference between non-chaotic dynamical systems and chaotic dynamical systems.

## 3. FINANCIAL SYSTEM

### 3.1. Dynamic of Financial System

Recently, the studies in (Ma and Chen, 2001) have reported a dynamic model of finance, composed of three first-order differential equations. The model describes the time-variation of three state variables: the interest rate  $x$ , the investment demand  $y$ , and the price index  $z$ . The factors that influence the changes of  $x$  mainly come from two aspects: firstly, it is the contradiction from the investment market, \_the surplus between investment and savings\_; secondly, it is the structure adjustment from goods prices. The changing rate of  $y$  is in proportion with the rate of investment, and in proportion by inversion with the cost of investment and the interest rate. The changes of  $z$ , on one hand, are controlled by the contradiction between supply and demand of the commercial market, and on the other hand, are influenced by the inflation rate. Here we suppose that the amount of supplies and demands of commercials is constant in a certain period of time, and that the amount of supplies and demands of commercials is in proportion by inversion with the prices. However, the changes of the inflation rate can in fact be represented by the changes of the real interest rate and the inflation rate equals the nominal interest rate subtracts the real interest rate. The original model has nine independent parameters to be adjusted, so it needs to be further simplified. Therefore, by choosing the appropriate coordinate system and setting an appropriate dimension to every state variable, we can get the following more simplified model with only three most important parameters:(Mohammed Salah Abd-Elouahab *et al.*, 2010).

$$\begin{aligned} D^{q_1} x &= z + (y_1 - a)x \\ D^{q_2} y &= 1 - by - x^2 \\ D^{q_3} z &= -x - cz \end{aligned} \quad (1)$$

Where  $a \geq 0$  the saving is amount,  $b \geq 0$  the cost per investment, and  $c \geq 0$  is the elasticity of demand of commercial markets. It is obvious that all three constants,  $a$ ,  $b$ , and  $c$ , are nonnegative.

## 4. ACTIVE SLIDING MODE CONTROLLER DESIGN AND ANALYSIS OF ECONOMIC SYSTEMS

### 4.1. Active Sliding Mode Controller Design

Consider a chaotic fractional-order system of order  $q(0 < q < 1)$  described by the following nonlinear fractional-order differential equation.

$$\begin{aligned} d^q x_1 &= z_1 + (y_1 - a)x_1 \\ d^q y_1 &= 1 - by_1 - x_1 x_1 \\ d^q z_1 &= -x_1 - cz \end{aligned} \quad (2)$$

The controller  $u(t) \in R^3$  is added into the slave system, so it is given by.

$$\begin{aligned} d^q x_2 &= z_2 + (y_2 - a)x_2 + u_1 \\ d^q y_2 &= 1 - by_2 - x_2 x_2 + u_2 \\ d^q z_2 &= -x_2 - cz_2 + u_3 \end{aligned} \quad (3)$$

Synchronization of the systems means finding a control signal  $u(t) \in R^3$  that makes states of the slave system to evolve as the states of the master system.

That the non-linear are system:

$$\begin{aligned} gx_1 &= x_2 y_2 - x_1 y_1 - \frac{x_1}{5} \\ gx_2 &= x_1^2 - x_2^2 - \frac{y_1}{100} \\ gx_3 &= -\frac{z_1}{20} \end{aligned} \quad (4)$$

Where:

$$\begin{aligned} e_1 &= x_1 - x_2 \\ e_2 &= y_1 - y_2 \\ e_3 &= z_1 - z_2 \end{aligned} \quad (5)$$

And  $G = gx_2 - gx_1 + (A_2 - A_1) \times [x_1, x_2, x_3]$  To simplify the notations, the linear part of the slave system

is represented by matrix  $A = A_2$  The aim is to design the controller  $u(t) \in R^3$  such

that:

$$\begin{aligned} \|e(t)\| &= 0 \\ t &\rightarrow \infty \end{aligned} \quad (6)$$

In accordance with the active control design procedure], the nonlinear part of the error dynamics is eliminated by the following choice of the input vector:

$$u = -k \times (c \times (r \times \text{diag}([111]) + A_2) \times [e_1, e_2, e_3] + p \times \text{sign}(s)) / (c \times k) - G \quad (7)$$

here  $k = (k_1, k_2, k_3)^T$  is a constant gain vector .

in which  $s = s(e)$  is a switching surface that prescribes the desired dynamics.

#### 4.2. Sliding surface design

The sliding surface can be defined as follows:

$$s(e) = ce \quad (8)$$

Where:

$$s = c(1) \times e_1 + c(2) \times e_2 + c(3) \times e_3 \quad (9)$$

Where,  $c = [c_1, c_2, c_3]$  is a constant vector. The equivalent control is found by the fact that  $\dot{s}(e) = 0$  is a necessary condition for the state trajectory to stay on the switching surface  $s(e) = 0$ . Hence, when in sliding mode, the controlled system satisfies the following conditions:

$$\dot{s}(e) = 0 \qquad s(e) = 0 \qquad (10)$$

.the error dynamics on the sliding surface are determined by the following relation:

$${}_0D_t^{1-q} e = A - k(ck)^{-1} c(rI + A) \qquad (11)$$

### 4.3. Design of the Sliding Mode Controller

We consider the constant plus proportional rate reaching law in our study (Zhang *et al.*, 2004; Haeri *et al.*, 2007). Then the reaching law is chosen as:

$${}_0D_t^{1-q} s = -\rho \operatorname{sgn}(s) - rs \qquad (12)$$

where  $\operatorname{sgn}(\cdot)$  denotes the sign function. The gains  $\rho > 0$  and  $r > 0$  are determined such that the sliding condition is satisfied and the sliding mode motion occurs.

## 5. NUMERICAL SIMULATIONS

The main aim of this paper is the designing based on the active sliding mode control for the synchronization of fractional-order chaotic systems. The chaos in the economic series could have serious and very different consequences in common macro-economy models. The controlled systems in this thesis are based on the framework of master – slave which was followed by a two different -stage design in this paper. At first, the active sliding mode controller has been designed for synchronization of chaotic systems of chaotic economic with different fractional order and it adds uncertainty to the system. Finally, the active sliding mode controller was designed to synchronize chaotic economic systems with different fractional order.

### 5.1. Simulation 1

Numerical simulations for chaotic economic system with different fractional-orders and with changing the system initial conditions and in the coefficients of the system are presented.

It is obvious that all three constants, a, b, and c, are nonnegative. Where The master system  $a=3, b=0.1, c=1$  and initial conditions  $(2,3,2)$  Is considered. Where The slave system  $a=3.2, b=1.01, c=1.05$  and initial conditions  $(3,4,3)$  Is considered. in which Synchronization is shown in Figures( 1,2,3), and error reach to minimize such that it is shown in fig(4,5,6).

Master system:

$$d^{0.84} x_1 = z_1 + (y_1 - a)x_1$$

$$d^{0.84} y_1 = 1 - by_1 - x_1x_1$$

$$d^{0.84} z_1 = -x_1 - cz_1$$

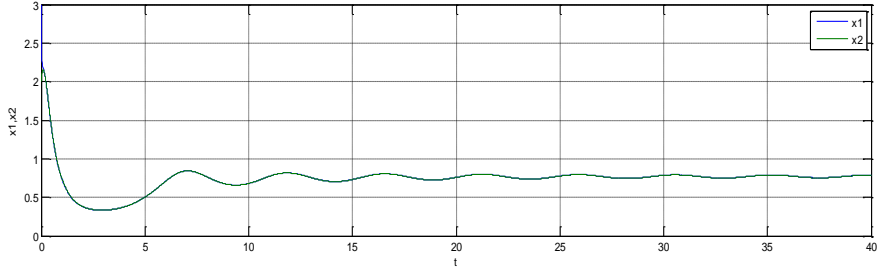
Slave system:

$$d^{0.84}x_2 = z_2 + (y_2 - a)x_2$$

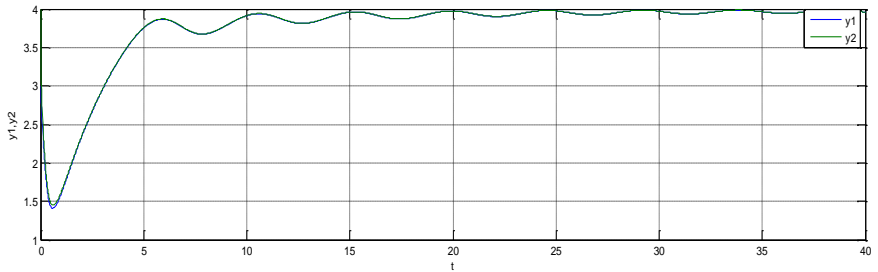
$$d^{0.82}y_2 = 1 - by_2 - x_2x_2$$

$$d^{0.80}z_2 = -x_2 - cz_2$$

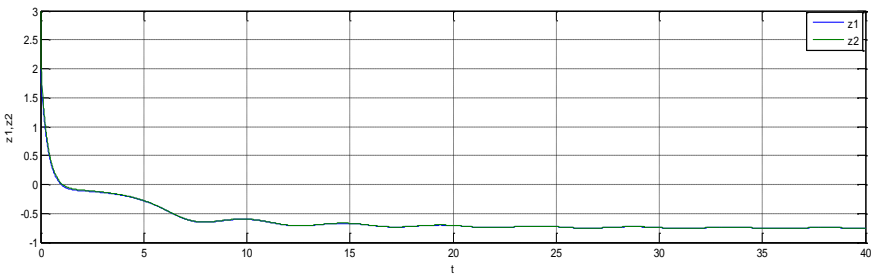
(Fig-1. Master and slave systems state synchronized trajectories [x1,x2] )



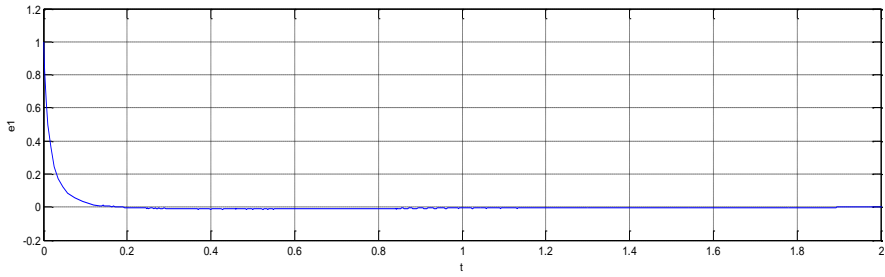
(Fig-2. Master and slave systems state synchronized trajectories [y1,y2] )



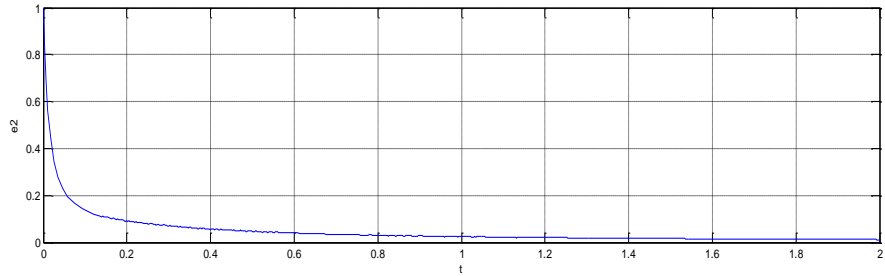
(Fig-3. Master and slave systems state synchronized trajectories [z1,z2] )



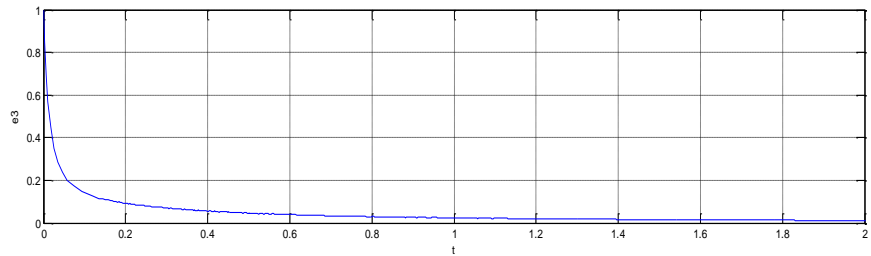
(Fig-4. Error trajectories of the synchronized master and slave systems [x1,x2])



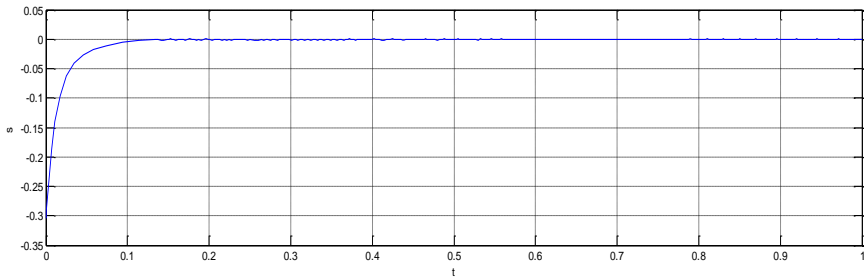
(Fig-5. Error trajectories of the synchronized master and slave systems [y1,y2])



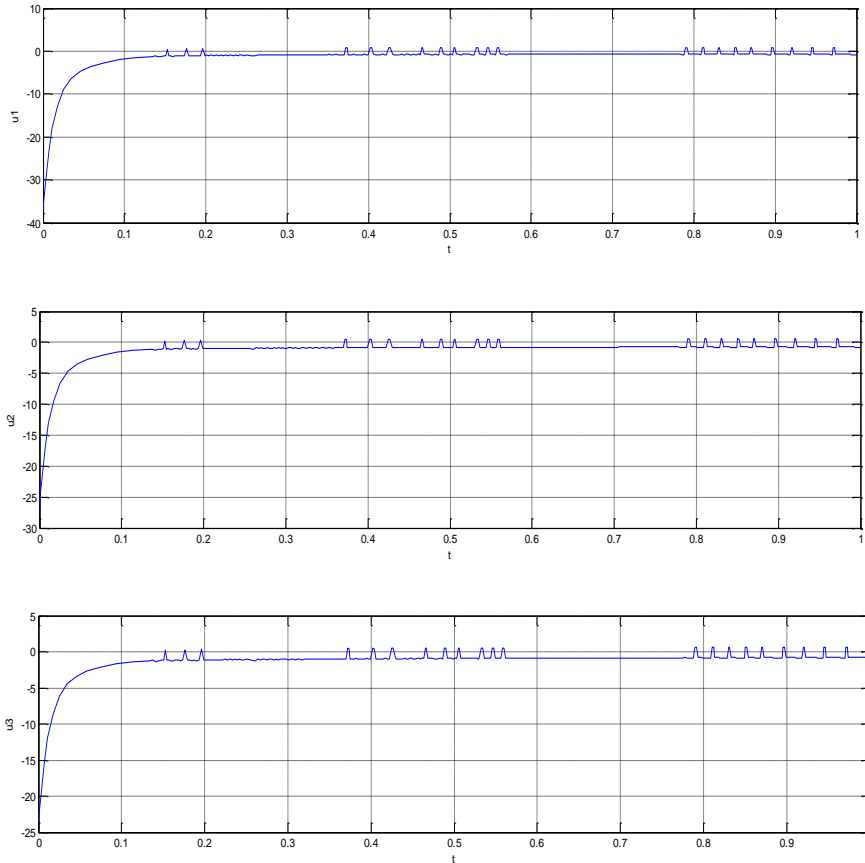
(Fig-6. Error trajectories of the synchronized master and slave systems [z1,z2])



(Fig-7. Sliding surface)



(Fig-8. Control signals)



## 5.2. Simulation 2

In this part the designing of active sliding mode controller for synchronization of chaotic economic system considering two different initial conditions and the 0.84 order is presented.

Where The master system  $a=3, b=0.1, c=1$  and initial conditions  $(2, 3, 2)$  Is considered .Where The slave system  $a=3, b=1., c=1$ , and initial conditions  $(3, 4, 3)$  is considered. In which Synchronization is shown in Figs. (9, 10, 11), and error reach to minimize such that it is shown in Figs. (12, 13, 14).

### Master system

$$d^{0.84} x_1 = z_1 + (y_1 - a)x_1$$

$$d^{0.84} y_1 = 1 - by_1 - x_1x_1$$

$$d^{0.84} z_1 = -x_1 - cz_1$$

### Slave system

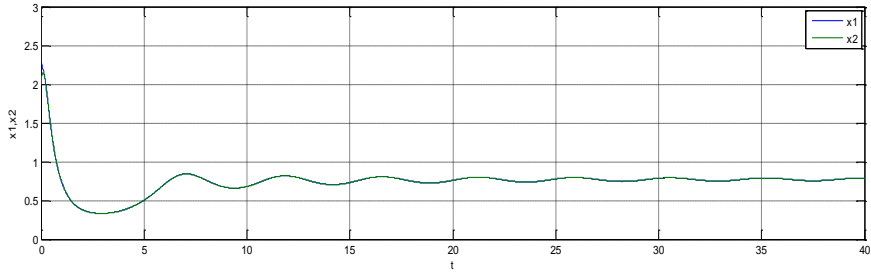
$$d^{0.84} x_2 = z_2 + (y_2 - a)x_2$$

$$d^{0.82} y_2 = 1 - by_2 - x_2x_2$$

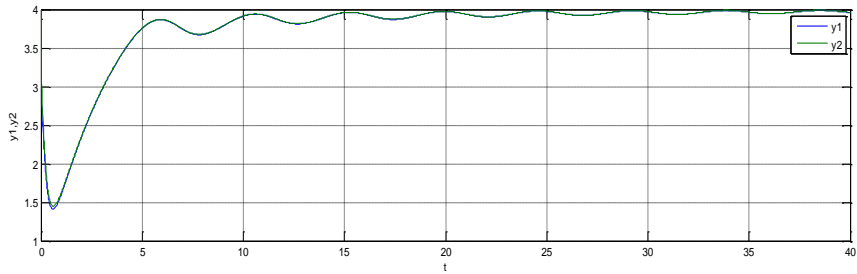
$$d^{0.80} z_2 = -x_2 - cz_2$$



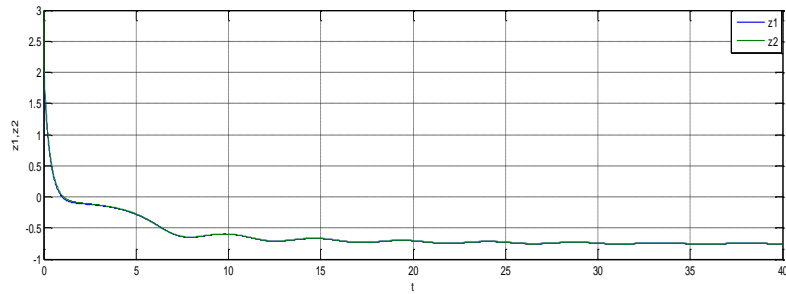
(Fig-9. Master and slave systems state synchronized trajectories [x1,x2] )



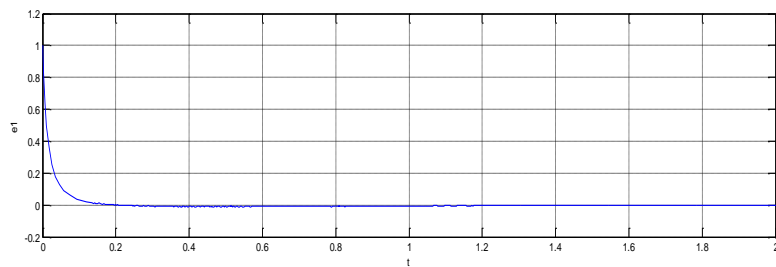
(Fig-10. Master and slave systems state synchronized trajectories [y1,y2] )



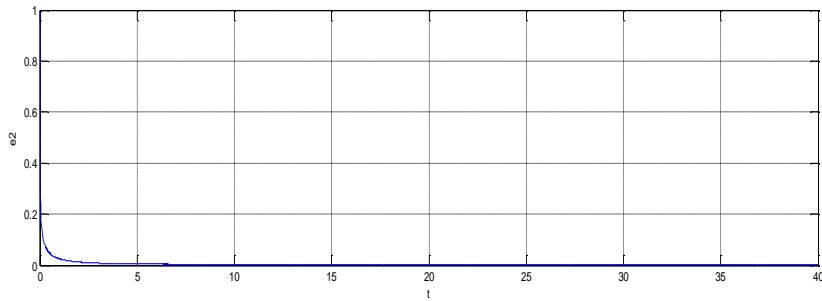
(Fig-11. Master and slave systems state synchronized trajectories [z1,z2] )



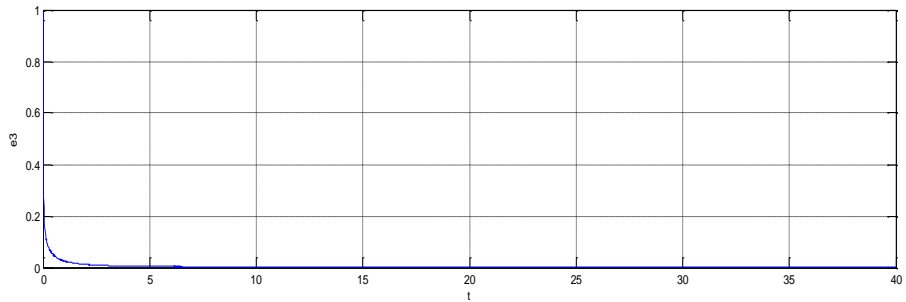
(Fig-12. Error trajectories of the synchronized master and slave systems [x1,x2])



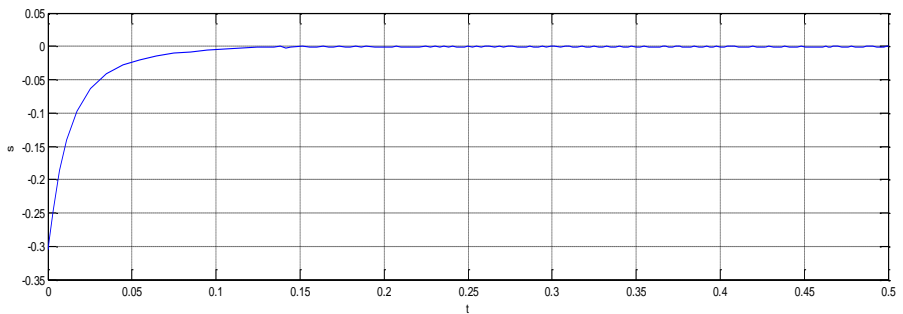
(Fig-13. Error trajectories of the synchronized master and slave systems [y1,y2])



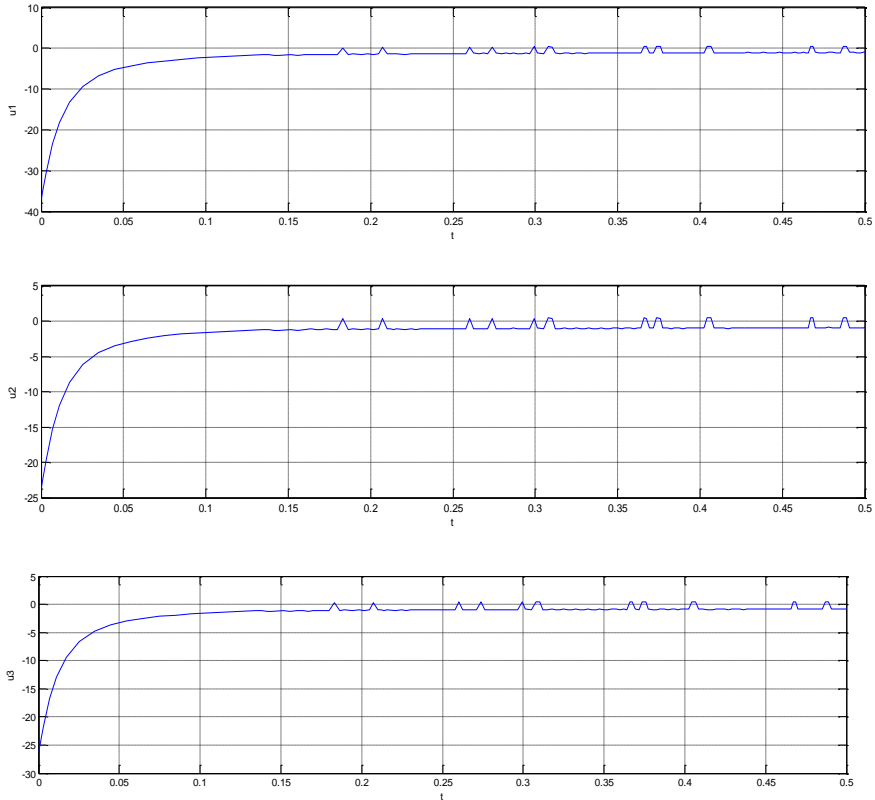
(Fig-14. Error trajectories of the synchronized master and slave systems [z1,z2])



(Fig-15. Sliding surface)



(Fig-16. Control signals)



## 6. CONCLUSIONS

In this article we proposed an active sliding mode control for synchronization of chaotic systems with different and the same orders in the form of master-slave systems. In addition, we provided a stability analysis of the proposed controller. Using active sliding mode control would be quite useful for systems with fractional derivatives. With the right choice of control parameters ( $r$ ,  $K$ ,  $C$ )-based and client (sequent) systems are synchronized. The sliding control nature improves the robustness of the controller. Trade between robustness and efficiency can be accomplished by altering the parameter  $p$ . Numerical simulation shows the usefulness of the proposed controller to synchronize the fractional-order chaotic systems. in which Synchronization is shown in some examples.

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