



## CLOSED FORM SOLUTION FOR HESTON PDE BY GEOMETRICAL TRANSFORMATIONS

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### ABSTRACT

*One presents and discusses an alternative solution writeable in closed form of the Heston’s PDE, for which the solution is known in literature, up to an inverse Fourier transform. Since the algorithm to compute the inverse Fourier Transform is not able to be applied easily for every payoff, one has elaborated a new methodology based on changing of variables which is independent of payoffs and does not need to use the inverse Fourier transform algorithm or numerical methods as Finite Difference and Monte Carlo simulation. In particular, one will compute the price of Vanilla Options for small maturities in order to validate numerically the Geometrical Transformations technique. The principal achievement is to use an analytical formula to compute the prices of derivatives, in order to manage, balance any portfolio through the Greeks, that by the proposed solution one is able to compute analytically. The above mentioned numerical techniques are computationally expensive in the stochastic volatility market models and for this reason is usually employed the Black Scholes model, that is unsuitable, as it has been widely proven in literature to compute the sensitivities of a portfolio and the price of derivatives. The present article wants to introduce a new approach to solve PDEs complicated, such as, those coming out from the stochastic volatility market models, with the achievement to reduce the computational cost and thus the time machine; besides, the proposed solution is easy to be generalized by adding jump processes as well. The present research work is rather technical and one does wide use of functional analysis. For the conceptual simplicity of the technique, one confides which many applications and studies will follow, extending the applications of the Geometrical Transformations technique to other derivative contracts of different styles and asset classes.*

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### 1. INTRODUCTION

The Black and Scholes model rests upon a number of assumptions that are, to some extent,

strategic. Among these there are the continuity of the stock price process (it does not jump), the ability to hedge continuously without transaction costs, independent Gaussian returns, and constant volatility. One is going to focus here on relaxing the last assumption by allowing volatility to vary randomly, for the following reason: a well known discrepancy between the Black and Scholes predicted European option prices and market traded options prices, the smile curve, can be accounted for by stochastic volatility models. Modelling volatility as a stochastic process is motivated a priori by empirical studies of the stock price returns in which estimated volatility is observed to exhibit random characteristics. Additionally, the effects of transaction costs show up, under many models, as uncertainty in the volatility; fat tailed returns distributions can be simulated by stochastic volatility. The assumption of constant volatility is not reasonable, since one requires different values for the volatility parameter for different strikes and different expiries to match market prices. The volatility parameter that is required in the Black Scholes formula to reproduce market prices is called the implied volatility. This is a critical internal inconsistency, since the implied volatility of the underlying should not be dependent on the specifications of the contract. Thus to obtain market prices of options maturing at a certain date, volatility needs to be a function of the strike. This function is the so called volatility skew or smile. Furthermore for a fixed strike one also needs to different volatility parameters to match the market prices of options maturing on different dates written on the same underlying, hence volatility is a function of both the strike and the expiry date of the derivative security. This bivariate function is called the volatility surface. There are two prominent ways of working around this problem, namely, local volatility models and stochastic volatility models. For local volatility models the assumption of constant volatility made in Black and Scholes is relaxed. The underlying risk-neutral stochastic process becomes:

$$dS_t = r(t)S_t dt + \sigma(t, S_t)S_t d\tilde{W}_t$$

where  $r(t)$  is the instantaneous forward rate of maturity  $t$  implied by the yield curve and the function  $\sigma(S_t, t)$  is chosen (calibrated) such that the model is consistent with market data. It is claimed in Hagan *et al.* (2002) that local volatility models predict that the smile shifts to higher prices (or lower prices) when the price of the underlying decreases (or increases). This is in contrast to the market behaviour where the smile shifts to higher prices (or lower prices) when the price of the underlying increases (or decreases). Another way of working around the inconsistency of introduced by constant volatility is by a stochastic process for the volatility itself; such models are called stochastic volatility models. The major advances in stochastic volatility models are in Hull and White (1987), Heston (1993) and Hagan *et al.* (2002). Generally speaking, stochastic volatility models are not complete, and thus a typical contingent claim (such as a European option) cannot be priced by arbitrage. In other words, the standard replication arguments can no longer be applied. For this reason, the issue of valuation of derivative securities under market incompleteness has attracted considerable attention in recent years, and various alternative approaches to this problem have been subsequently developed. Seen from a different perspective, the incompleteness of a generic stochastic volatility model is reflected by the fact that the class of all martingale measures

for the process  $S_t/B_t$  comprises more than one probability measure, and thus there is the necessity to specify a single pricing probability measure. For this purpose, one needs to first specify the market price of volatility risk  $\lambda(v, t)$ . Mathematically speaking, the market price for the risk is associated to the drift of stochastic processes and it can be changed by the Girsanov's theorem. Let us observe that the price of volatility risk  $\lambda(v, t)$  has to satisfy the Feller condition, such that the volatility process is nonnegative, and Hason (2010).

## 2. HESTON PDE AND ITS TRANSFORMATIONS

In this section one discusses a series of coordinate transformations in order to reduce the Heston PDE in a simpler. Assume that the dynamic of the couple of diffusion processes  $(S_t, \nu_t)$ , under a martingale measure, is given by (1), that is the famous Heston market model:

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{\nu_t} S_t d\tilde{W}_t^{(1)} \\ d\nu_t &= \kappa(\theta - \nu_t) dt + \alpha \sqrt{\nu_t} d\tilde{W}_t^{(2)} \\ f(T, S_T, \nu_T) &= \Phi(S_T) \end{aligned} \tag{1}$$

where  $E_Q[dW_t^{(1)} dW_t^{(2)}] = \rho dt$  for some constant  $\rho \in [-1, +1]$ , and suppose also that both processes  $S_t$  and  $\nu_t$  are nonnegative.

By Ito's lemma one has the two-dimensional PDE (2), and for solving it, suitable numerical procedures need to be employed. The calculations based on the discretization of the partial differential equation satisfied by the pricing function appear excessively time consuming. An alternative Monte Carlo approach for stochastic volatility models has been examined by Fourier approach, but also in this case one has an excessively time consuming. Other techniques are introduced from Forde and Jacquier (2009), Avramidi (2010) and Dell'Era (2010), which are not without problems.

Following the idea to reduce the Heston's PDE in another simpler, by the change of variables, see [12], one can obtain an alternative solution which could be useful for its generality.

One shows the proposed method in what follows: one writes the partial differential equation for pricing options, assuming that (1) is the market model and  $f(T, S_T, \nu_T)$  is the payoff of a derivative contract:

$$\begin{aligned} \frac{\partial f}{\partial t} + \frac{1}{2} \nu \left( S^2 \frac{\partial^2 f}{\partial S^2} + 2\rho\alpha S \frac{\partial^2 f}{\partial S \partial \nu} + \alpha^2 \frac{\partial^2 f}{\partial \nu^2} \right) + rS \frac{\partial f}{\partial S} + \kappa(\theta - \nu) \frac{\partial f}{\partial \nu} - rf &= 0 \\ f(T, S, \nu) &= \Phi_0(S_T), \quad \rho \in (-1, +1), \quad \alpha \in \mathbb{R}^+ \\ S \in [0, +\infty) \quad \nu \in [0, +\infty) \quad t \in [0, T] \end{aligned} \tag{2}$$

where  $\Phi(S_T)$  is a general payoff function for a derivative security. Consider some coordinate transformations in the right order:

$$1^{st} \text{ transformation} \begin{cases} x = \ln S, & x \in (-\infty, +\infty) \\ \tilde{\nu} = \nu/\alpha, & \tilde{\nu} \in [0, +\infty) \\ f(t, S, \nu) = f_1(t, x, \tilde{\nu})e^{-r(T-t)} \end{cases}$$

thus one has:

$$\frac{\partial f_1}{\partial t} + \frac{1}{2}\tilde{\nu} \left( \frac{\partial^2 f_1}{\partial x^2} + 2\rho \frac{\partial^2 f_1}{\partial x \partial \tilde{\nu}} + \frac{\partial^2 f_1}{\partial \tilde{\nu}^2} \right) + \left( r - \frac{1}{2}\alpha\tilde{\nu} \right) \frac{\partial f_1}{\partial x} + \frac{\kappa}{\alpha} (\theta - \alpha\tilde{\nu}) \frac{\partial f_1}{\partial \tilde{\nu}} = 0$$

$$f_1(T, x, \tilde{\nu}) = \Phi_1(x) \quad \rho \in (-1, +1), \quad \alpha \in \mathbb{R}^+$$

$$x \in (-\infty, +\infty) \quad \tilde{\nu} \in [0, +\infty) \quad t \in [0, T]$$

Again, one makes another coordinates transformation:

$$2^{nd} \text{ transformation} \begin{cases} \xi = x - \rho\tilde{\nu} & \xi \in (-\infty, +\infty) \\ \eta = -\tilde{\nu}\sqrt{1-\rho^2} & \eta \in (-\infty, 0] \\ f_1(t, x, \tilde{\nu}) = f_2(t, \xi, \eta) \end{cases}$$

and one has:

$$\frac{\partial f_2}{\partial t} - \frac{\alpha\eta}{2\sqrt{1-\rho^2}}(1-\rho^2) \left( \frac{\partial^2 f_2}{\partial \xi^2} + \frac{\partial^2 f_2}{\partial \eta^2} \right) + \frac{\alpha\eta}{2\sqrt{1-\rho^2}} \left( 1 - \frac{2\kappa\rho}{\alpha} \right) \frac{\partial f_2}{\partial \xi} - \frac{\alpha\eta}{2\sqrt{1-\rho^2}} \left( \frac{2\kappa}{\alpha}\sqrt{1-\rho^2} \right) \frac{\partial f_2}{\partial \eta}$$

$$+ \left( r - \frac{\kappa\rho\theta}{\alpha} \right) \frac{\partial f_2}{\partial \xi} - \frac{\theta\kappa}{\alpha}\sqrt{1-\rho^2} \frac{\partial f_2}{\partial \eta} = 0$$

$$f_2(T, \xi, \eta) = \Phi_2(\xi, \eta) \quad \rho \in (-1, +1), \quad \alpha \in \mathbb{R}^+$$

$$\xi \in (-\infty, +\infty) \quad \eta \in (-\infty, 0] \quad t \in [0, T].$$

Finally, by the third coordinates transformation:

$$3^{rd} \text{ transformation} \begin{cases} \gamma = \xi + \left( r - \frac{\kappa\rho\theta}{\alpha} \right) (T-t) & \gamma \in (-\infty, +\infty) \\ \phi = -\eta + \frac{\kappa\theta}{\alpha}\sqrt{1-\rho^2}(T-t) & \phi \in [0, +\infty) \\ \tau = \frac{1}{2} \int_t^T \nu_s ds & \tau \in [0, +\infty) \\ f_2(t, \xi, \eta) = f_3(\tau, \gamma, \phi) \end{cases}$$

$f_2(t, \xi, \eta) = f_3(\tau, \gamma, \phi)$  where  $\gamma = \gamma(t, \xi)$ ,  $\phi = \phi(t, \eta)$  and  $\tau$  is a random variable, function of the variable  $t$ , for the Fundamental Theorem of Integral Calculus.

That being said, by the third transformation one obtains the following PDE:

$$\frac{\partial f_3}{\partial \tau} = (1 - \rho^2) \left( \frac{\partial^2 f_3}{\partial \gamma^2} + \frac{\partial^2 f_3}{\partial \phi^2} \right) - \left( 1 - \frac{2\kappa\rho}{\alpha} \right) \frac{\partial f_3}{\partial \gamma} - \left( \frac{2\kappa}{\alpha} \sqrt{1 - \rho^2} \right) \frac{\partial f_3}{\partial \phi} = 0$$

(3)

which is simpler than (2). Imposing that:  $f_3(\tau, \gamma, \phi) = e^{a\tau + b\gamma + c\phi} f_4(\tau, \gamma, \phi)$ , where:

$$\begin{cases} a = -(1 - \rho^2)(b^2 + c^2); \\ b = \frac{(1 - \frac{2\kappa\rho}{\alpha})}{2(1 - \rho^2)}; \\ c = \frac{\kappa}{\alpha\sqrt{1 - \rho^2}}; \end{cases}$$

one has:

$$\begin{aligned} \frac{\partial f_4}{\partial \tau} &= (1 - \rho^2) \left( \frac{\partial^2 f_4}{\partial \gamma^2} + \frac{\partial^2 f_4}{\partial \phi^2} \right) \\ f_4(0, \gamma, \phi) &= \Phi_4(\gamma, \phi) \\ \tau \in [0, +\infty) \quad \phi \in [0, +\infty) \quad \gamma \in (-\infty, +\infty). \end{aligned}$$

(4)

The solution of the PDE (4) is known in the literature (Andrei D. Polyanin, Handbook of Linear Partial Differential Equations, 2002, p. 188), and it can be written as integral, whose kernel  $G(0, \gamma', \phi' | \tau, \gamma, \phi)$  is a bivariate gaussian function:

$$G(0, \gamma', \phi' | \tau, \gamma, \phi) = \frac{1}{4\pi\tau(1 - \rho^2)} \left[ e^{-\frac{(\gamma' - \gamma)^2 + (\phi' - \phi)^2}{4\tau(1 - \rho^2)}} - e^{-\frac{(\gamma' - \gamma)^2 + (\phi' + \phi)^2}{4\tau(1 - \rho^2)}} \right],$$

therefore

$$\begin{aligned} f_4(\tau, \gamma, \phi) &= \int_0^{+\infty} d\phi' \int_{-\infty}^{+\infty} d\gamma' f_4(0, \gamma', \phi') G(0, \gamma', \phi' | \tau, \gamma, \phi) \\ &\quad + (1 - \rho^2) \int_0^\tau du \int_{-\infty}^{+\infty} d\gamma' f_4(u, \gamma', 0) \left[ \frac{\partial G(0, \gamma', \phi' | \tau - u, \gamma, \phi)}{\partial \phi'} \right]_{\phi'=0}, \end{aligned}$$

namely since  $f(t, S, \nu) = e^{-r(T-t) + a\tau + b\gamma + c\phi} f_4(\tau, \gamma, \phi)$ , one has:

$$f(t, S, \nu) = e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^{+\infty} d\phi' \int_{-\infty}^{+\infty} d\gamma' f_4(0, \gamma', \phi') G(0, \gamma', \phi' | \tau, \gamma, \phi) + (1 - \rho^2) e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^\tau du \int_{-\infty}^{+\infty} d\gamma' f_4(u, \gamma', 0) \left[ \frac{\partial G(0, \gamma', \phi' | \tau - u, \gamma, \phi)}{\partial \phi'} \right]_{\phi'=0},$$

and

$$f(t, S, \nu) = e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^{+\infty} d\phi' \int_{-\infty}^{+\infty} d\gamma' \Phi_4(\gamma', \phi') G(0, \gamma', \phi' | \tau, \gamma, \phi) + (1 - \rho^2) e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^\tau du \int_{-\infty}^{+\infty} d\gamma' f_4(u, \gamma', 0) \left[ \frac{\partial G(0, \gamma', \phi' | \tau - u, \gamma, \phi)}{\partial \phi'} \right]_{\phi'=0}, \tag{5}$$

where  $\Phi_4$  is the payoff written in the new variables  $(\gamma, \phi)$ , see (4) and  $f_4(u, \gamma, 0)$  is the value of the option over the time for  $\phi = 0$ , that should be simulated; however it is not interesting and in what follows, one is going to consider only small maturities, such that the second term of the last integral goes to zero, since  $\tau \rightarrow 0$  when  $T \rightarrow 0$ .

### 3. VANILLA OPTION PRICING

In order to test above option pricing formula (5), one considers as option, a Vanilla Call with strike price  $K$  and maturity  $T$ . In the new variable the payoff  $(S_T - K)^+$  is equal to  $e^{-b\gamma - c\phi} (e^{\gamma + \rho\phi / \sqrt{1 - \rho^2}} - K)^+$ . Substituting this latter in the equation (5) one has:

$$f(t, S, \nu) = e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^{+\infty} d\phi' \int_{-\infty}^{+\infty} d\gamma' e^{-b\gamma' - c\phi'} (e^{\gamma' + \rho\phi' / \sqrt{1 - \rho^2}} - K)^+ G(0, \gamma', \phi' | \tau, \gamma, \phi) + (1 - \rho^2) e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^\tau du \int_{-\infty}^{+\infty} d\gamma' f_4(u, \gamma', 0) \left[ \frac{\partial G(0, \gamma', \phi' | \tau - u, \gamma, \phi)}{\partial \phi'} \right]_{\phi'=0}. \tag{6}$$

Avoiding all tedious computations, and using the stochastic notation, the solution is:

$$f(t, S_t, \nu_t) = S_t \left[ \mathbb{N} \left( -\psi_1(0), -a_{1,1} \sqrt{1 - \rho^2} \right) - e^{-2(\rho - \frac{\kappa}{\alpha}) \left( \frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha} \theta (T-t) \right)} \mathbb{N} \left( -\psi_2(0), -a_{1,2} \sqrt{1 - \rho^2} \right) \right] - K e^{-r(T-t)} \left[ \mathbb{N} \left( -\tilde{\psi}_1(0), -a_{2,1} \sqrt{1 - \rho^2} \right) - e^{2\frac{\kappa}{\alpha} \left( \frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha} \theta (T-t) \right)} \mathbb{N} \left( -\tilde{\psi}_2(0), -a_{2,2} \sqrt{1 - \rho^2} \right) \right] + (1 - \rho^2) e^{-r(T-t)+a\tau+b\gamma+c\phi} \int_0^\tau du \int_{-\infty}^{+\infty} d\gamma' f_4(u, \gamma', 0) \left[ \frac{\partial G(0, \gamma', \phi' | \tau - u, \gamma, \phi)}{\partial \phi'} \right]_{\phi'=0}, \tag{7}$$

where the term  $f_4(u, \gamma, \phi=0)$  is the value over the time, of the considered Option, written using the new variables  $(\gamma, \phi)$ , whereas  $u$  is an integration variable such as  $\gamma'$ .

The value of a Vanilla Option, by the solution (7) becomes, for small maturities ( $T \rightarrow 0$ ), equal to:

$$f(t, S_t, \nu_t) = S_t \left[ \mathbb{N} \left( -\psi_1(0), -a_{1,1} \sqrt{1 - \rho^2} \right) - e^{-2(\rho - \frac{\kappa}{\alpha})(\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha} \theta(T-t))} \mathbb{N} \left( -\psi_2(0), -a_{1,2} \sqrt{1 - \rho^2} \right) \right] - K e^{-r(T-t)} \left[ \mathbb{N} \left( -\tilde{\psi}_1(0), -a_{2,1} \sqrt{1 - \rho^2} \right) - e^{2\frac{\kappa}{\alpha}(\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha} \theta(T-t))} \mathbb{N} \left( -\tilde{\psi}_2(0), -a_{2,2} \sqrt{1 - \rho^2} \right) \right], \tag{8}$$

and this is true because for  $T \rightarrow 0$  also  $\tau \rightarrow 0$ , and the second integral of (7) is intuitively worthless.

So that, the solution (8) can be considered as an approximation for brief maturities of the general solution (7).

#### 4. NUMERICAL VALIDATION

In what follows one considers the approximation  $\tau \rightarrow 0$  which will be interpreted as the price of an Option with few days to maturity. From 1 day up to 10 days are suitable maturities to prove the validation hypothesis, at varying of the volatility. The chosen parameter values are those in Bakshi *et al.* (1997) namely  $\kappa = 1.15$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$  and  $\rho = -0.64$  where the spot interest rate is  $r = 10\%$ , and strike price is  $K = \$100$ , on the three different maturities  $T$  above indicated. It is worth noting that in all function arguments of the solution (8):  $\psi_1(0)$ ,  $\psi_2(0)$ ,  $a_{1,1}$ ,  $a_{1,2}$ ,  $a_{2,1}$ ,  $a_{2,2}$ , there is the term:

$$\frac{1}{2} \int_t^T \nu_s ds$$

(see Appendix), which needs to be computed numerically, since  $\nu_s$  is the variance process (1). However for large maturities, one can use the Ergodic theorem, for which:

$$\frac{1}{T} \int_0^T \nu_s ds \rightarrow \mathbb{E}_{\mathbb{P}}[\nu_T]$$

that is known analytically.

In the table hereafter one can see the results of numerical experiments:

**Table 1: At the money**,  $S_0 = 100$ ,  $K = 100$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and **Maturity 1 day**.

Volatility	Fourier method	Dell’Era method
30%	0.6434	0.6442
40%	0.8543	0.8541
50%	1.0643	1.0641
60%	1.2743	1.2742
70%	1.4843	1.4845
80%	1.6943	1.6949
90%	1.9042	1.9055
100%	2.1142	2.1162

Table 2: At the money,  $S_0 = 100$ ,  $K = 100$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 5 days.

Volatility	Fourier method	Dell'Era method
30%	1.4763	1.4748
40%	1.9430	1.9407
50%	2.4101	2.4081
60%	2.8772	2.8769
70%	3.3444	3.3472
80%	3.8115	3.8190
90%	4.2785	4.2927
100%	4.7454	4.7683

Table 3: At the money,  $S_0 = 100$ ,  $K = 100$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 10 days.

Volatility	Fourier method	Dell'Era method
30%	2.1234	2.1191
40%	2.7787	2.7722
50%	3.4348	3.4294
60%	4.0912	4.0905
70%	4.7477	4.7557
80%	5.4040	5.4254
90%	6.0601	6.1002
100%	6.7158	6.7806

Table 4: In the money,  $S_0 = K \left(1 + 10\% \sqrt{\theta(T-t)}\right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 1 day.

Volatility	Fourier method	Dell'Era method
30%	0.6991	0.6994
40%	0.9094	0.9089
50%	1.1191	1.1187
60%	1.3289	1.3287
70%	1.5377	1.5389
80%	1.7488	1.7494
90%	1.9588	1.9600
100%	2.1688	2.1708



Table 5: In the money,  $S_0 = K \left(1 + 10\% \sqrt{\theta(T-t)}\right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 5 days.

Volatility	Fourier method	Dell'Era method
30%	1.6049	1.6012
40%	2.0700	2.0661
50%	2.5362	2.5331
60%	3.0030	3.0019
70%	3.4700	3.4723
80%	3.9372	3.9445
90%	4.4044	4.4186
100%	4.8715	4.8947

Table 6: In the money,  $S_0 = K \left(1 + 10\% \sqrt{\theta(T-t)}\right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 10 days.

Volatility	Fourier method	Dell'Era method
30%	2.3098	2.3012
40%	2.9621	2.9527
50%	3.6168	3.6095
60%	4.2727	4.2708
70%	4.9291	4.9366
80%	5.5856	5.6072
90%	6.2421	6.2831
100%	6.8984	6.9647

Table 7: Out the money,  $S_0 = K \left(1 - 10\% \sqrt{\theta(T-t)}\right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 1 day.

Volatility	Fourier method	Dell'Era method
30%	0.5905	0.5918
40%	0.8013	0.8014
50%	1.0111	1.0112
60%	1.2210	1.2212
70%	1.4309	1.4313
80%	1.6407	1.6415
90%	1.8506	1.8519
100%	2.0605	2.0625

Table 8: Out the money,  $S_0 = K \left( 1 - 10\% \sqrt{\theta(T-t)} \right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 5 days.

Volatility	Fourier method	Dell'Era method
30%	1.3539	1.3546
40%	1.8208	1.8201
50%	2.2878	2.2869
60%	2.7546	2.7551
70%	3.2214	3.2247
80%	3.6882	3.6959
90%	4.1547	4.1689
100%	4.6212	4.6438

Table 9: Out the money,  $S_0 = K \left( 1 - 10\% \sqrt{\theta(T-t)} \right)$ , with parameter values:  $\kappa = 1.5$ ,  $\theta = 0.04$ ,  $\alpha = 0.39$ ,  $\rho = -0.64$ ,  $r = 0.10$  and Maturity 10 days.

Volatility	Fourier method	Dell'Era method
30%	1.9459	1.9459
40%	2.6019	2.5985
50%	3.2581	3.2548
60%	3.9142	3.9148
70%	4.5701	4.5787
80%	5.2257	5.2471
90%	5.8810	5.9204
100%	6.5359	6.5992

### 5. GREEKS

Given a portfolio composed by equity assets, bond and derivatives, as for example European Call and Put Options, the most great problem that one has to deal is to manage the variation over the time of its value as well as to compute its daily risk exposition, by the Value at Risk techniques (VaR, CVaR, etc.). In all these cases one needs to compute the sensitivities, known in literature as Greeks. These latter are defined by the following derivatives:

$$\frac{\partial f(t, S, \nu, r)}{\partial S}, \frac{\partial^2 f(t, S, \nu, r)}{\partial^2 S}, \frac{\partial f(t, S, \nu, r)}{\partial \nu}, \frac{\partial f(t, S, \nu, r)}{\partial r}, \frac{\partial f(t, S, \nu, r)}{\partial t}, \frac{\partial^2 f(t, S, \nu, r)}{\partial S \partial \nu}, \frac{\partial^2 f(t, S, \nu, r)}{\partial \nu^2}$$

by which one can describe the sensitivities of a portfolio, to the variations of the value of held derivatives, with respect to the variations of the variables:  $S_t$  spot price,  $\nu_t$  variance,  $(T-t)$  time to maturity and  $r$  spot interest rate.

As one has just seen in the last section, the formula (8) computes right prices for small maturities of Vanilla Options, but it can be used also for other derivatives, thus to compute the sensitivities of a portfolio is rather simple, since one is able to compute these analytically by the formula (8) as:

$$\begin{aligned}\Delta &= \frac{\partial f(t, S, \nu, r)}{\partial S} \\ &= \frac{\partial}{\partial S} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}\Gamma &= \frac{\partial^2 f(t, S, \nu, r)}{\partial S^2} \\ &= \frac{\partial^2}{\partial S^2} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}V &= \frac{\partial f(t, S, \nu, r)}{\partial \nu} \\ &= \frac{\partial}{\partial \nu} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}\Theta &= \frac{\partial f(t, S, \nu, r)}{\partial t} \\ &= \frac{\partial}{\partial t} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}\rho &= \frac{\partial f(t, S, \nu, r)}{\partial r} \\ &= \frac{\partial}{\partial r} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}V_\nu &= \frac{\partial^2 f(t, S, \nu, r)}{\partial \nu^2} \\ &= \frac{\partial^2}{\partial \nu^2} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\ &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\tilde{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha} + \frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\tilde{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],\end{aligned}$$

$$\begin{aligned}
 V_{S,\nu} &= \frac{\partial f(t, S, \nu, r)}{\partial S \partial \nu} \\
 &= \frac{\partial}{\partial S \partial \nu} \left[ S_t \left( \mathbb{N}(-\psi_1(0), -a_{1,1}\sqrt{1-\rho^2}) - e^{-2(\rho-\frac{\sigma}{\alpha})(\frac{\nu}{\alpha}+\frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\psi_2(0), -a_{1,2}\sqrt{1-\rho^2}) \right) \right. \\
 &\quad \left. - K e^{-r(T-t)} \left( \mathbb{N}(-\bar{\psi}_1(0), -a_{2,1}\sqrt{1-\rho^2}) - e^{2\frac{\sigma}{\alpha}(\frac{\nu}{\alpha}+\frac{\sigma}{\alpha}\theta(T-t))} \mathbb{N}(-\bar{\psi}_2(0), -a_{2,2}\sqrt{1-\rho^2}) \right) \right],
 \end{aligned}$$

in this way, one can obtain a great computational advantage, together to the absence of numerical errors, due to numerical techniques.

Usually, one uses Monte Carlo simulation method to compute Greeks by evaluating numerically the following expected value:

$$\frac{\partial f(t, S, \nu, r)}{\partial S} = \mathbb{E}_{\mathbb{P}} \left[ \frac{\phi(T-t, S + \delta S, \nu, r) - \phi(T-t, S, \nu, r)}{\delta S} \right],$$

(9)

where  $\phi$  is the payoff and  $\phi(T-t, S, \nu, r)$  is the intrinsic value at time  $t$  of the derivative. As one can image from (9), to evaluate the sensitivities of a whole portfolio is really heavy from computational viewpoint and therefore to reduce the computational cost it is essential to manage better the wealth of a portfolio composed by several derivatives correlated to each other.

## 6. CONCLUSIONS

The main problem that one has using the Fourier's technique for pricing options, is that there is any possibility to calculate the numeric error directly, but one needs to compare Fourier prices with Monte Carlo prices, for which one can manage the variance, establishing the error; besides the Fast Fourier Transform algorithm (known as FFT ) used to calculate the anti-Fourier transform for Vanilla Options, is not easy to generalize to other derivatives as well. For these reasons find a new approach to solve Heston's PDE is a sensible argument in Finance, as shown by the wide literature, in which one can find many articles in matter (see the References section). The proposed method is straightforward from theoretical viewpoint and it is independent to the payoff and therefore, to price derivatives have the same algorithmic complexity for every payoff, unlike using Fourier Transform method, for which the complexity is tied to the payoff. By the introduced methodology, one reduces the Heston's PDE in a simpler, using, in a right order, some suitable changing of variables, whose Jacobian has not singularity points, unless for  $\rho = \pm 1$  (this evidence gives the safety that the chosen variables are well defined). The PDE (4), which is the last transformed PDE, is an heat equation, whose solution is known in literature and it gives the price of options in closed form, as proven in the section Vanilla Option Pricing, by the equation (7). In the present paper one has discussed only a particular case, for small maturities, and the numerical results are satisfactory. But small maturities are sufficient to compute Greeks, that are an important instrument to manage and balance any portfolio composed with derivatives. For this reason one can think to use the proposed technique is better than Heston's solution to evaluate VaR, CVaR and so on. The numerical

evidence gives the boost to spend other time in this research direction, in order to generalize the methodology for every maturity and payoff. The present article wants to be a new approach to solving PDEs complicated as come out from the Heston market model, for which the proposed solution is easy to generalize; with adding for example of jump processes. Therefore for the simplicity of the technique, one confides which many applications and studies will follow.

## 7. APPENDIX

$$\psi_1(0) = -\frac{\left[\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha}\theta(T-t) + (\rho - \frac{\kappa}{\alpha}) \int_t^T \nu_s ds\right]}{\sqrt{\int_t^T \nu_s ds}},$$

$$\psi_2(0) = \frac{\left[\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha}\theta(T-t) - (\rho - \frac{\kappa}{\alpha}) \int_t^T \nu_s ds\right]}{\sqrt{\int_t^T \nu_s ds}},$$

$$\tilde{\psi}_1(0) = -\frac{\left[\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha}\theta(T-t) - \frac{\kappa}{\alpha} \int_t^T \nu_s ds\right]}{\sqrt{\int_t^T \nu_s ds}},$$

$$\tilde{\psi}_2(0) = \frac{\left[\frac{\nu_t}{\alpha} + \frac{\kappa}{\alpha}\theta(T-t) + \frac{\kappa}{\alpha} \int_t^T \nu_s ds\right]}{\sqrt{\int_t^T \nu_s ds}},$$

$$a_{1,1} = \frac{\left[\ln(K/S_t) - r(T-t) - \frac{1}{2} \int_t^T \nu_s ds\right]}{\sqrt{(1-\rho^2) \int_t^T \nu_s ds}},$$

$$a_{1,2} = \frac{\left[\ln(K/S_t) + 2\frac{\rho}{\alpha}\nu_t - (r - 2\frac{\kappa\rho}{\alpha})(T-t) - \frac{1}{2} \int_t^T \nu_s ds\right]}{\sqrt{(1-\rho^2) \int_t^T \nu_s ds}},$$

$$a_{2,1} = \frac{\left[\ln(K/S_t) - r(T-t) + \frac{1}{2} \int_t^T \nu_s ds\right]}{\sqrt{(1-\rho^2) \int_t^T \nu_s ds}},$$

$$a_{2,2} = \frac{\left[\ln(K/S_t) + 2\frac{\rho}{\alpha}\nu_t - (r - 2\frac{\kappa\rho}{\alpha})(T-t) + \frac{1}{2} \int_t^T \nu_s ds\right]}{\sqrt{(1-\rho^2) \int_t^T \nu_s ds}}.$$

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