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A MODEL FOR ESTIMATING THE DISTRIBUTION OF FUTURE POPULATION

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ABSTRACT

In most counties, statistical authorities collect data on the number of deaths in each age group. These data enables the calculation of life expectancy as well as the calculation of death and survival probabilities for each age group. In this paper, we develop an easy-to-use tool to estimate the distribution of future survivors for each cohort. Such a distribution defines the probabilities for the number of survivors at a given future time. Many institutions can benefit from the estimation of the distribution of future survivors (e.g., pension funds, geriatric institutions and medical authorities in general). Thus, our paper contributes not only to the literature on the projection of mortality rates, but it also has significant practical implications.

Keywords: Distribution, Future survivors, Life expectancy, Cohort, Mortality rates, Projection.

Contribution/ Originality

This study uses new methodology for estimating future distribution of survivors. Estimating the probability of N_T surviving individuals in time T via the continuous normal distribution solution requires a multiple-integral calculation. We suggest a compact model that allows doing the calculation via a single integral over the density distribution function.

1. INTRODUCTION

In most counties, statistical authorities collect data on the number of deaths in each age group, which enables the calculation of death probabilities (and survival probabilities) as well as the calculation of life expectancy for each age group. However, a large degree of uncertainty is associated with these future survival probabilities.

Booth and Tickle (2008) reviewed the history of mortality projections and identified three main projection methodologies. The first methodology builds epidemiological models in order to explain particular causes of death or known risk factors (see, Alderson and Ashwood, 1985). The second methodology defines a target life expectancy and then determines a path to that life expectancy. Such an approach allows expert opinion to be factored into the statistical process (see Pollard, 1987; Olshansky, 1988). The third methodology is extrapolation - a term given to the forecasting of future mortality based on past patterns (see Lee and Carter, 1992; Milevsky and Promislow, 2001; Brouhns *et al.*, 2002; Dahl, 2004; Biffis, 2005; Renshaw and Haberman, 2006; Schrager, 2006; Cairns *et al.*, 2009).

Many institutions would be interested in and benefit from the estimation of the distribution of future population. Such a distribution defines the probabilities for the number of survivors for each cohort at any given time in the future.

First of all, pension funds would benefit from forecasting the distribution of the expected number of survivors in cohorts passed retirement age. When a death of an insured individual occurs, pension funds usually pay a given compensation to the family in addition to the continuation of the post-death pension payments (payment level might be different relative to the pre-death payments). Funds can use past records and documentations of death probabilities in order to evaluate the number of expected deaths. However, there is a high probability that the actual number of deaths would be much larger than the anticipated number (calculated by the multiplication of the number of people in a cohort by their probability of dying). This means that even if pension funds operate with an expected actuarial balance, they might end up extremely unbalanced due to volatility in the expected forthcoming deaths. If pension funds could forecast the distribution of the number of deaths for each cohort, it would improve their forecasting tools (by using the confidence interval of the expected number of deaths instead of solely using the expected value) and thus will enable them to take steps to reduce their operational risks.

Insurance companies would also find it useful to evaluate the distribution of the future expected number of survivors for each cohort since they are interested in estimating the probabilities associated with their commitments, i.e., upcoming death compensation payments. Although death probabilities enable the estimation of expected payments, they do not fully describe the complete risks. For example, if the probability of dying for a given cohort is 2%, then the expected number of deaths for 1,000 individuals (from this cohort) is 20. However, there is some probability that more (and even much more) than 20 individuals will die, causing substantial losses to the insurance company. The same considerations hold for deaths caused by car accidents (drivers' insurance risks), earthquake insurance risks as well as health insurance risks.

Additionally, medical authorities would be interested in the same information. For example, medical authorities evaluate death probabilities caused by epidemic diseases based on past records. In particular, they are interested in the probability that the actual number of deaths would be much larger than the expected number of deaths. Moreover, being exposed to a disease might create a given risk of death in all future periods. Here again, the medical authority would benefit from taking into consideration the distribution of the future deaths in each time period.

A variety of approaches have been proposed for modeling randomness in the aggregation of mortality rates over time. Lee and Carter (1992) worked with discrete time models and focused on periodical application of stochastic mortality and its statistical analysis (see also Renshaw and Haberman, 2006). Other researchers developed continuous time frameworks (see Dahl, 2004; Schrager, 2006).

In this paper, we first assume that in each future period an agent faces a given likelihood to survive. We assume that future annual survival probability is defined according to the report of the realized death probabilities published by the statistical authorities. Then, we present a model that enables the prediction of future survival distribution for each cohort.

2. THEORY/CALCULATION

2.1. Estimating the Distribution of Future Survivors for Each Cohort

2.1.1. The Bernoulli Distribution and the Binomial Distribution

Let us assume that N_0 individuals were born in period 0. According to statistical authorities, their survival probability during period 1 is p_1. That probability can be found in mortality rates tables published by the authorities. We can refer to the survival distribution at age 1 as a Bernoulli distribution. Each agent faces a "successful trial" with probability p_1 (survival) and a failure with a probability $1-p_1$ (death). Using the Newton Binomial formula, we can calculate the probability $prob_1(j)$ that j agents will survive at age t=1, for j = 0,1,2,…N_0. As $N_0 \to \infty$, the binomial distribution function is expressed in terms of the standard normal distribution function.

Using the same idea, the distribution of the number of survivors in period t depends on the realized number of survivors in period t-1, N_{t-1}. Thus, given the distribution of the number of survivors in time t-1, we can calculate the distribution of the number of survivors in the following period. Each individual faces a survival probability p_t during period t, and, given that N_{t-1} individuals survived period t-1, any j surviving agents face a survival probability $prob_t(j\,|\,N_{t-1})$ in time t. The surviving agents face in time t+1 a new individual survival probability p_{t+1} and, given that N_t individuals survived period t, a new Bernoulli trial with survival probability $prob_{t+1}(j\,|\,N_t)$.

2.1.2. An Example: The Binomial Distribution

In order to simplify the presentation, let us assume that we estimate the distribution of survival of a nonhuman species with survival probabilities of p_1 and p_2 at years 1 and 2, respectively. Assuming that only N=3 objects were born in year zero, table 1 below shows the calculations of the future survival distribution of the objects born in year 0, according to the Bernoulli distribution.

Table-1-Panel-A. The survival distribution in year 1 and in year 2 (given the number of survivors in year 1)

Number of survivors in year 1	Year 1 survival probability	Number of survivors in year 2	Year 2 survival Probability given the number of survivors in year 1
0	$\binom{3}{0}p_1^{\,0}(1-p_1)^3$	0	1
1	$\binom{3}{1}p_1^{\,1}(1-p_1)^2$	0	$\binom{1}{0}p_2^{\,0}(1-p_2)^1$
		1	$\binom{1}{1}(p_2^{\,1}(1-p_2)^0$
2	$\binom{3}{2}p_1^{\,2}(1-p_1)^1$	0	$\binom{2}{0}p_2^{\,0}(1-p_2)^2$
		1	$\binom{2}{1}p_2^{\,1}(1-p_2)^1$
		2	$\binom{2}{2}p_2^{\,2}(1-p_2)^0$
3	$\binom{3}{3}p_1^{\,3}(1-p_1)^0$	0	$\binom{3}{0}p_2^{\,0}(1-p_2)^3$
		1	$\binom{3}{1}p_2^{\,1}(1-p_2)^2$
		2	$\binom{3}{2}p_2^{\,2}(1-p_2)^1$
		3	$\binom{3}{3}p_2^{\,3}(1-p_2)^0$

Notice that the survival probability in year 2 (table 1 - panel B) is the product of the probability of j survivors in year 1, j=0,1,2,3, multiplied by the probability of k survivors in year 2, such that k is lower than or equal to j. For example, the probability that j=2 objects survive year 1 and k=1 object survives year 2 is:

$$\binom{3}{2}p_1^{\,2}(1-p_1)^1 * \binom{2}{1}p_2^{\,1}(1-p_2)^1$$

where:

$$\binom{3}{2} p_1^{\ 2}(1-p_1)^1$$

is the probability that 2 people survive year 1 and

$$\binom{2}{1} p_2^{\ 1}(1-p_2)$$

is the probability that 1 person survives year 2 given that 2 people had survived year 1.

Based on table 1, the overall survival distributions in years 1 and 2 are presented in table 2. The year 1 survival probabilities are similar to those presented in panel A of table 1. The year 2 survival

Table-1–Panel-B. Survival distribution in year 2

Number of survivors in year 1	Number of survivors in year 2	Year 2 overall survival probability given the number of survivors in year 1
0	0	$\binom{3}{0} p_1^{\ 0}(1-p_1)^3$
1	0	$\binom{3}{1} p_1^{\ 1}(1-p_1)^2 * \binom{1}{0} p_2^{\ 0}(1-p_2)^1$
	1	$\binom{3}{1} p_1^{\ 1}(1-p_1)^2 * \binom{1}{1} p_2^{\ 1}(1-p_2)^0$
2	0	$\binom{3}{2} p_1^{\ 2}(1-p_1)^1 * \binom{2}{0} p_2^{\ 0}(1-p_2)^2$
	1	$\binom{3}{2} p_1^{\ 2}(1-p_1)^1 * \binom{2}{1} p_2^{\ 1}(1-p_2)^1$
	2	$\binom{3}{2} p_1^{\ 2}(1-p_1)^1 * \binom{2}{2} p_2^{\ 2}(1-p_2)^0$
3	0	$\binom{3}{3} p_1^{\ 3}(1-p_1)^0 * \binom{3}{0} p_2^{\ 0}(1-p_2)^3$
	1	$\binom{3}{3} p_1^{\ 3}(1-p_1)^0 * \binom{3}{1} p_2^{\ 1}(1-p_2)^2$
	2	$\binom{3}{3} p_1^{\ 3}(1-p_1)^0 * \binom{3}{2} p_2^{\ 2}(1-p_2)^1$
	3	$\binom{3}{3} p_1^{\ 3}(1-p_1)^0 * \binom{3}{3} p_2^{\ 3}(1-p_2)^0$

probabilities are calculated as follows. For each possible number of survivors k (k=0,1,2,3) in period t=2, we take into consideration all possible number of survivors j in the preceding period (time 1), $j \geq k \qquad j = k,...N_0$, and then we sum these probabilities of the different scenarios leading to k survivors in time 2:

$$p(k) = \sum_{j=k}^{N_0} p(j) * p(k|j) \cdot$$

To explain table 2, consider year 2 survival probability for one survival:

$$(1)\quad \underbrace{\binom{3}{1}p_1^{\,1}(1-p_1)^2}_{\substack{\text{Prob of one survivor}\\ \text{in period 1}}} * \binom{1}{1}p_2^{\,1}(1-p_2)^0 + \underbrace{\binom{3}{2}p_1^{\,2}(1-p_1)^1}_{\substack{\text{Prob of 2 survivors}\\ \text{in period 1}}} * \binom{2}{1}p_2^{\,1}(1-p_2)^1 + \underbrace{\binom{3}{3}p_1^{\,3}(1-p_1)^0}_{\substack{\text{Prob of 3 survivors}\\ \text{in period 1}}} * \binom{3}{1}p_2^{\,1}(1-p_2)^2$$

2.1.3. An Example: The Binomial Distribution – A Numerical Solution

Let us now continue the binomial distribution example from section 1.1.1. As an example, given $p_1 = 0.96$ and $p_2 = \frac{2}{3}$, the survival distribution in years 1 and 2 are presented in table 3.

Table-2. The overall Survival distribution in years 1 and 2

Number of survivals	Year 1 survival probability	Year 2 survival probability
0	$\binom{3}{0}p_1^{\,0}(1-p_1)^3$	$\binom{3}{0}p_1^{\,0}(1-p_1)^3 + \binom{3}{1}p_1^{\,1}(1-p_1)^2 * \binom{1}{0}p_2^{\,0}(1-p_2)^1 + \binom{3}{2}p_1^{\,2}(1-p_1)^1 * \binom{2}{0}p_2^{\,0}(1-p_2)^2 + \binom{3}{3}p_1^{\,3}(1-p_1)^0 * \binom{3}{0}p_2^{\,0}(1-p_2)^3$
1	$\binom{3}{1}p_1^{\,1}(1-p_1)^2$	$\binom{3}{1}p_1^{\,1}(1-p_1)^2 * \binom{1}{1}p_2^{\,1}(1-p_2)^0 + \binom{3}{2}p_1^{\,2}(1-p_1)^1 * \binom{2}{1}p_2^{\,1}(1-p_2)^1 + \binom{3}{3}p_1^{\,3}(1-p_1)^0 * \binom{3}{1}p_2^{\,1}(1-p_2)^2$
2	$\binom{3}{2}p_1^{\,2}(1-p_1)^1$	$\binom{3}{2}p_1^{\,2}(1-p_1)^1 * \binom{2}{2}p_2^{\,2}(1-p_2)^0 + \binom{3}{3}p_1^{\,3}(1-p_1)^0 * \binom{3}{2}p_2^{\,2}(1-p_2)^1$
3	$\binom{3}{3}p_1^{\,3}(1-p_1)^0$	$\binom{3}{3}p_1^{\,3}(1-p_1)^0 * \binom{3}{3}p_2^{\,3}(1-p_2)^0$

Table-3. The survival distribution in years 1 and 2 given that $p_1 = 0.96$ and $p_2 = \frac{2}{3}$

Number of survivors	Year 1 survival probability	Year 2 survival probability
0	6.40E-05	0.046656
1	0.004608	0.248832
2	0.110592	0.442368
3	0.884736	0.262144

2.2. The Normal Distribution

In order to ease the calculations, we can use the principle that as $N_t \rightarrow \infty$, the binomial distribution function is expressed in terms of the standard normal distribution function.

Suppose N_{t-1} is the number of surviving individuals at the end of time t-1, p_t is the probability of surviving the upcoming period t and X_t is a random variable representing the number of surviving individuals at the end of the period t. Then $X_t \sim B(N_{t-1}, p_t)$ and as $N_{t-1} \rightarrow \infty$,$X_t \sim N(N_{t-1}p_t, N_{t-1}p_t(1-p_t))$.

We define:

$\emptyset()$ – the PDF (density distribution function) for the normal distribution.

N_0 – initial number of individuals in time 0.

N_T – the required number of surviving individuals in time T.

X_t – a random variable, normally distributed, represents the number of surviving individuals in time t.

x_t – the realized number of surviving individuals in time t.

p_t- the probability of surviving in time t.

The implementation of these definitions yields that X_1 represents the number of surviving individuals in time 1 and is normally distributed:

$$(4)\ X_1 \sim N\left(N_0 p_1, N_0 p_1(1-p_1)\right)$$

In general, the normal distribution for X_t depends on x_{t-1} – the realized number of surviving individuals in time t-1. For t=2,3,…,T, this distribution of X_t given x_{t-1} is:

$$(5)\ X_t|X_{t-1} \sim N\left(x_{t-1}p_t, x_{t-1}p_t(1-p_t)\right).$$

The PDF of $X_1$is:

$$(6)\ \emptyset(X_1 = x_1) = \frac{1}{\sqrt{2\pi N_0 p_1(1-p_1)}} e^{\frac{-(x_1 - N_0 p_1)^2}{2N_0 p_1(1-p_1)}},$$

while the PDF of $X_t|X_{t-1}$ is:

$$(7)\ \emptyset(X_t = x_t|X_{t-1} = x_{t-1}) = \frac{1}{\sqrt{2\pi x_{t-1} p_t(1-p_t)}} e^{\frac{-(x_t - x_{t-1} p_t)^2}{2x_{t-1} p_t(1-p_t)}}.$$

For simplicity, we start with a simple example, the same as the example discussed in sections 1.1.1-1.1.2. Let us describe the probability for 1 surviving individual in time 2. We substitute T=2 and $N_T = 1$ and get that:

$$(8)\ X_1 \sim N\left(N_0 p_1, N_0 p_1(1-p_1)\right)$$

$$(9)\ X_2|X_1 \sim N\left(x_1 p_2, x_1 p_2(1-p_2)\right).$$

Thus, the probability for 1 surviving individual in time 2 is:

$$(10) \int_{0.5}^{N_0+0.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1.$$

Note that equation (10) includes two terms: the external integral represents the probability for the number of survivors in time 1, and the internal integral represents the probability that only one individual will survive in period 2. Thus, we can further subdivide equation (10) into:

$$(11) \int_{0.5}^{N_0+0.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1 = \int_{0.5}^{1.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1 + \int_{1.5}^{2.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1 + \int_{2.5}^{3.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1 + \cdots + \int_{N_0-1.5}^{N_0-0.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1 + \int_{N_0-0.5}^{N_0+0.5} \emptyset(X_1) \int_{0.5}^{1.5} \emptyset(X_2|X_1 = x_1) \partial x_2 \partial x_1$$

Given that $N_0 = 3$, $p_1 = 0.96$ and $p_2 = \frac{2}{3}$, we use equation (10) for calculating the survival distributions in year 2, and the results are as follows: the probability for 1 survivor in period 2 is 0.26449, the probability for 2 survivors in period 2 is 0.437144, and the probability for 3 survivors in period 2 is 0.184538. These numbers are similar in magnitude to those presented in Table 3. The differences can be explained by the small sample of $N_0 = 3$ individuals.

Generalizing equation (10), we can express the probability of N_T surviving individuals in time T=2 as:

$$(12) \int_{N_T-0.5}^{N_0+0.5} \emptyset(X_1) \int_{N_T-0.5}^{N_T+0.5} \emptyset(X_2|X_1 = x_1)\partial x_2 \partial x_1.$$

Placing the PDFs of $X_2|X_1 = x_1$ and of X_1, as defined by equations (6)-(9), into equation (12), we get:

$$(13) \int_{N_T-0.5}^{N_0+0.5} \emptyset(X_1) \int_{N_T-0.5}^{N_T+0.5} \emptyset(X_2|X_1 = x_1)\partial x_2 \partial x_1 =$$

$$\int_{N_T-0.5}^{N_0+0.5} \frac{1}{\sqrt{2\pi N_0 p_1(1-p_1)}} e^{\frac{-(x_1-N_0p_1)^2}{2N_0p_1(1-p_1)}} \int_{N_T-0.5}^{N_T+0.5} \frac{1}{\sqrt{2\pi x_1 p_2(1-p_2)}} e^{\frac{-(x_2-x_1p_2)^2}{2x_1p_2(1-p_2)}} \partial x_2 \partial x_1.$$

Note that after the internal integration over X_2 in equation (13), we get an expression that depends on X_1. Then, when integrating the external integration over X_1, it does the integration also over the X_1 that remained after the internal integration.

Now let us generalize the example above to T time periods and N_T survivors in time T. We get that the probability of N_T survivors in time T is:

$$(14) \int_{N_T-0.5}^{N_0+0.5} \emptyset(X_1 = x_1) \int_{N_T-0.5}^{x_1+0.5} \emptyset(X_2 = x_2|X_1 = x_1) \ldots$$

$$\int_{N_T-0.5}^{x_{t-1}+0.5} \emptyset(X_t = x_t|X_{t-1} = x_{t-1}) \ldots \int_{N_T-0.5}^{x_{T-2}+0.5} \emptyset(X_{T-1} = x_{T-1}|X_{T-2} = x_{T-2})$$

$$\int_{N_T-0.5}^{N_T+0.5} \emptyset(X_T = x_T|X_{T-1} = x_{T-1})\partial x_T \partial x_{T-1} \ldots \partial x_t \ldots \partial x_2 \partial x_1$$

Let us explain the underlying intuition of equation (14). The first part of equation (14) is: $\int_{N_T-0.5}^{N_0+0.5} \emptyset(X_1 = x_1)$. x_1 is the realized number of surviving individuals in time 1. We would like x_1 to be lower than or equal to N_0 (since N_0 is the initial number of individuals in time 0, we cannot have more surviving individuals in time 1 than the initial number of individuals in the time 0), but greater than or equal to N_T (since N_T is the required number of surviving individuals in time T, we cannot have less surviving individuals in time 1).

The second part of equation (14) is: $\int_{N_T-0.5}^{x_1+0.5} \emptyset(X_2 = x_2|X_1 = x_1)$. We know now how many individuals survived time 1: the answer is x_1. Now we would like to consider the possible values for x_2 - the realized number of surviving individuals in time 2. We would like x_2 to be lower than or equal to x_1 (we cannot have more surviving individuals in time 2 than the number of surviving individuals in the time 1, the preceding period), but greater than or equal to $\mathrm{N_T}$ (since $\mathrm{N_T}$ is the required number of surviving individuals in time T, T>2, we cannot have less surviving individuals in time 2).

In general, we can explain what happens in time t. Part number t of equation (14) is: $\int_{\mathrm{N_T}-0.5}^{x_{t-1}+0.5} \emptyset(X_t = x_t | X_{t-1} = x_{t-1})$. We know now how many individuals survived time t-1: the answer is $\mathrm{x_{t-1}}$. Now we would like to consider the possible values for x_t - the realized number of surviving individuals in time t. We would like $\mathrm{x_t}$ to be lower than or equal to x_{t-1} (we cannot have more surviving individuals in time t than the number of surviving individuals in the time t-1), but greater than or equal to $\mathrm{N_T}$ (since $\mathrm{N_T}$ is the required number of surviving individuals in time T, T>t, we cannot have less surviving individuals in time t).

The last part of equation (14) is: $\int_{N_T-0.5}^{N_T+0.5} \emptyset(X_T = x_T | X_{T-1} = x_{T-1})$. We know now how many individuals survived time T-1: the answer is $\mathrm{x_{T-1}}$. We also know that x_T - the realized number of surviving individuals in time T – should equal to $\mathrm{N_T}$. Thus, the limits of the last integral of equation (14) are: $\mathrm{N_T} - 0.5$ and $\mathrm{N_T} + 0.5$. The result is the estimation of the probability of $\mathrm{N_T}$ surviving individuals in time T.

2.2.1. Another Example

We start with $\mathrm{N_0} = 4$ individuals in time 0. Let us describe the probability for 3 surviving individual in time 3. We substitute $\mathrm{N_0} = 4$, T=3 and $\mathrm{N_T} = 3$ and get that:

$$X_1 \sim N\big(N_0 p_1, N_0 p_1(1-p_1)\big),\ X_2|X_1 \sim N\big(x_1 p_2, x_1 p_2(1-p_2)\big),$$
$$X_3|X_2 \sim N\big(x_2 p_3, x_2 p_3(1-p_3)\big),\ X_4|X_3 \sim N\big(x_3 p_4, x_3 p_4(1-p_4)\big).$$

Thus, given that $N_0 = 4$, the probability for 3 surviving individual in time 3 is:

$$(15)\int_{2.5}^{4.5} \emptyset(X_1) \int_{2.5}^{x_1+0.5} \emptyset(X_2|X_1 = x_1) \int_{2.5}^{x_2+0.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3) \partial x_4 \partial x_3\, \partial x_2 \partial x_1.$$

Let us explain the underlying intuition of equation (15). The first part of equation (15) is: $\int_{2.5}^{4.5} \emptyset(X_1)$. $\mathrm{x_1}$ is the realized number of surviving individuals in time 1. We would like $\mathrm{x_1}$ to be lower than or equal to $\mathrm{N_0} = 4$ but greater than or equal to $\mathrm{N_3} = 3$. Explanation: notice that since we start with $\mathrm{N_0} = 4$, the initial number of individuals in time 0, we cannot have more surviving individuals in time 1. On the other hand, since $\mathrm{N_3} = 3$ is the required number of surviving individuals in time 3, we cannot have less than $\mathrm{N_3} = 3$ surviving individuals in time 1.

The second part of equation (15) is: $\int_{2.5}^{x_1+0.5} \emptyset(X_2 | X_1 = x_1)$. In time t=2, $\mathrm{x_1}$- the number of individuals that survived period 1 - is given and known. Now we would like to consider the possible values for $\mathrm{x_2}$ - the realized number of surviving individuals in time 2. We would like $\mathrm{x_2}$ to be lower than or equal to x_1 but greater than or equal to $\mathrm{N_3} = 3$. Explanation: since x_1 is the realized number of surviving individuals in time 1, we cannot have more surviving individuals in time 2. On the other hand, since $N_3 = 3$ is the required number of surviving individuals in time 3, we cannot have less surviving individuals in time 2.

The explanation for the third part of equation (15) is similar.

The fourth and last part of equation (15) is: $\int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)$. We know that x_T - the realized number of surviving individuals in time t – should be equal to $N_3 = 3$. Thus, the limits of this last part of equation (6) are: $3 - 0.5$ and $3 + 0.5$. The result is the estimation of $N_3 = 3$ surviving individuals in time 3.

In equation (16) we further decomposed equation (15) based on the following different scenarios:

1).$x_1 = 3, x_2 = 3, x_3 = 3, x_4 = 3$ (this is the first line in the decomposition of equation (15).

2). $x_1 = 4, x_2 = 3, x_3 = 3, x_4 = 3$ (this is the second line in the decomposition of equation (15).

3). $x_1 = 4, x_2 = 4, x_3 = 3, x_4 = 3$ (this is the third line in the decomposition of equation (15).

4). $x_1 = 4, x_2 = 4, x_3 = 4, x_4 = 3$ (this is the fourth line in the decomposition of equation (15).

$$(16) \int_{2.5}^{4.5} \emptyset(X_1) \int_{2.5}^{x_1+0.5} \emptyset(X_2|X_1 = x_1) \int_{2.5}^{x_2+0.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)\partial x_4 \partial x_3\, \partial x_2 \partial x_1$$

$$=$$

$$\int_{2.5}^{3.5} \emptyset(X_1) \int_{2.5}^{3.5} \emptyset(X_2|X_1 = x_1) \int_{2.5}^{3.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)\partial x_4 \partial x_3\, \partial x_2 \partial x_1 +$$

$$\int_{3.5}^{4.5} \emptyset(X_1) \int_{2.5}^{3.5} \emptyset(X_2|X_1 = x_1) \int_{2.5}^{3.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)\partial x_4 \partial x_3\, \partial x_2 \partial x_1 +$$

$$\int_{3.5}^{4.5} \emptyset(X_1) \int_{3.5}^{4.5} \emptyset(X_2|X_1 = x_1) \int_{2.5}^{3.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)\partial x_4 \partial x_3\, \partial x_2 \partial x_1 +$$

$$\int_{3.5}^{4.5} \emptyset(X_1) \int_{3.5}^{4.5} \emptyset(X_2|X_1 = x_1) \int_{3.5}^{4.5} \emptyset(X_3|X_2 = x_2) \int_{2.5}^{3.5} \emptyset(X_4|X_3 = x_3)\partial x_4 \partial x_3\, \partial x_2 \partial x_1.$$

2.3. Normal Distribution – The Compact Solution

Equation (14) in section 1.2 represents the continuous normal distribution solution. It estimates the probability of N_T surviving individuals in time T via T integrals, since it considers and accounts for every realization regarding the number of surviving individuals that could occur. The first integral goes over the possible values for the realized number of surviving individuals in time 1; The second integral goes over the possible values for the realized number of surviving individuals in time 2, given the realized number of survivors in period 1; Integral number t goes over the possible values for the realized number of surviving individuals in time t, given the realized number of survivors in period t-1; And the last integral, integral number T, estimates the probability for exactly N_T, the required number of surviving individuals in time T, given the realized number of survivors in period T-1.

Let us reconsider the role of the first T-1 integrals in equation (14). These T-1 integrals go over all possible values for the realized number of surviving individuals in times 1,2,3,…,T-1. The result of these T-1 integrals is explicit and simple: it is the calculation of the expected number of surviving individuals in time T-1.

In order to avoid solving the problem via the multiple–integral solution as in equation (14), we suggest another way to express the average expected number of individuals in time T-1, $\mathrm{N}_{\mathrm{T}-1}$.

We define the probability to survive from time 0 and until time T-1 as:

$$(16)\ \ p_{0\to T-1} = \prod_{i=1}^{T-1} p_i$$

Thus, the expected number of individuals at time T-1 is:

$$(17)\ \ \mathrm{N}_{\mathrm{T}-1} = N_0 * p_{0\to T-1.}$$

We can now express the probability of N_{T} surviving individuals in time T as:

$$(18) \int_{N_T-0.5}^{N_T+0.5} \emptyset(X_T)\partial x_T,$$

where:

$$(19)\ \ X_T \sim N\big(N_0 * p_{0\to T-1} * p_T, N_0 * p_{T0\to T-1} * p_T * (1-p_T)\big)$$

Using equations (18) and (19) instead of using the multiple integrals in equation (14) enable us to calculate the survival probability distribution at time T in a much more compact way.

2.3.1. A Real-Life Example

We used the mortality rates tables published by the United States Social Security Administration (SSA) to get the mortality rates for individuals born in 1960 – for each year between 1960 and 2007 (the survival rate is 1 minus the mortality rate). Table A.1 in appendix A presents part of the SSA data.

According to the data in table A.1, the probability of individuals born in 1960 to survive 1960 is 0.970626, the probability of those individuals, born in 1960, to survive 1961 is 0.998263, while the probability of those individuals to survive 1962 is 0.998949 and so on.

We can now calculate the probability of individuals that were born in 1960 to survive until 2006 (included) as the multiplication of the annual survival rates presented in table A.1 for the time period 1960 - 2006. The calculated probability of individuals born in 1960 to survive until 2006 (included) is 0.89195. Thus, among every 1000 individuals born in 1960, 891.947 will survive till the end of 2006, on average. The probability of those individuals surviving 2007 is also given in table A.1 and is equal to 0.99579.

The distribution of surviving individuals in 2007 is calculated via the following equation (20).The probability for N_{T} surviving individuals is -

$$(20) \int_{N_T-0.5}^{N_T+0.5} \emptyset(X_T)\partial x_T$$

for:

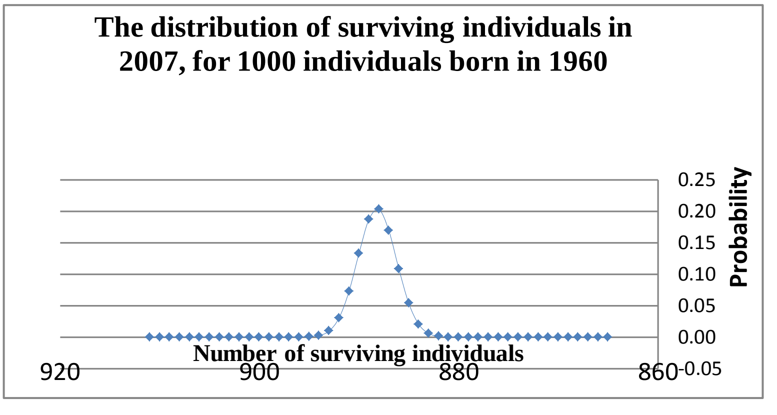
$$(21)\ X_T \sim N\big(891.947 * 0.99579, 891.947 * 0.99579 * (1-0.99579)\big).$$

We program the solution defined by the integral in equation (20) with its characteristics in (21) via the Wolfram Mathematica software. The Mathematica code is provided in appendix B.

Table C.1 in appendix C reports the distribution of surviving individuals in 2007. The distribution of surviving individuals in 2007 is also described in figure 1.

We can also forecast the estimated distribution of future survivors in any period in the future. For example, we can forecast the distribution of survivors in 2015 for the same group of individuals born in 1960. Let us go back to the SSA mortality tables which reports mortality rates only until the year 2007. The cohort born in 1960 was 47 years old in 2007. However, we can use the mortality rates data collected by the statistical agencies for 2007 for people at ages 48, 49, 50, 51, 52, 53, 54 and 55, and attribute them to the cohort born in 1960 and use it as estimation for the 2008, 2009, 2010, 2011, 2012, 2013, 2014, 2015 mortality probabilities, respectively. For example, the mortality rate for the cohort at the age of 48 in 2007 is 0.004603. Thus, we can estimate the forecasted 2008 mortality probability for the cohort born in 1960 as 0.004603 and thus we can estimate the forecasted survival probability (that equals 1 minus the forecasted mortality probability) as 0.995397. We continue to make these estimations until we reach the 2015 mortality probability forecast. These forecasts are presented in table D.1 in appendix D.

Figure-1.



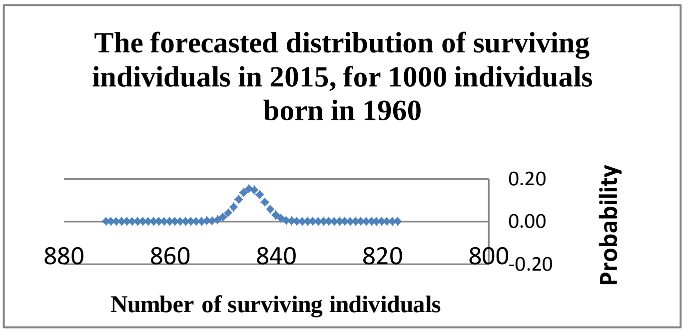
We showed before that for 1000 people born in 1960, the calculated probability of surviving until 2006 (included) was 0.89195 and thus the expected number of survivors in 2006 was 891.947. We can now calculate the probability of individuals born in 1960 to survive until 2014 (included) by multiplying the calculated probability of surviving until 2006 by the probability of surviving 2007 (given in the SSA table as well as in table D.1) and by the forecasted probabilities of surviving 2008-2014 (included), as reported in table D.1.

The estimated probability to survive until 2014 (included) is 0.85147 and thus the expected number of survivors in 2014 is 851.471 (out of the original 1000 individuals born in 1960). Using our technique and the forecasted survival probability for 2015 (which is 0.99203 – as reported in table D.1), we can calculate the expected distribution of future survivors in 2015. The forecasted distribution of future surviving individuals in 2015 is again calculated via equation (20), for:

$$(22)\ X_T \sim N(851.471 * 0.99203, 851.471 * 0.99203 * (1 - 0.99203)).$$

The forecasted distribution of the future survivors in 2015 is described in figure 2.

Figure-2.



3. RESULTS AND DISCUSSION

In most counties, the statistical authorities collect data on the number of deaths in each age group. That enables the calculation of life expectancy as well as the calculation of death and survival probabilities for each age group. However, many institutions (e.g., pension funds, geriatric institutions and the medical authorities in general) would benefit from estimating the future distribution of survivors as well.

In this paper, we develop a tool that can be used to estimate the future distribution of survivors for each cohort. Such a distribution defines the probability for the number of survivors at a given future time. Assuming N_0 individuals were born in period 0, and assuming that their probability to survive time t is defined as p_t, we can refer to the survival distribution at time t as a Bernoulli distribution. In the Bernoulli distribution, each agent faces a "successful trial" with probability p_t (survival) and a failure with the probability $1-p_t$ (death). Using the Newton Binomial formula we can calculate the probability that N_T individuals will survive at any time T>t. As $N_t \to \infty$, the binomial distribution function is expressed in terms of the standard normal distribution function.

Estimating the probability of N_T surviving individuals in time T via the continuous normal distribution solution requires a multiple-integral calculation. Alternatively, we suggest a compact model in which the average expected number of individuals in time T-1, N_{T-1}, is estimated by multiplying the probability to survive from time 0 until time T-1 by the initial number of individuals in time 0, N_0. That allows us to express the probability of N_T surviving individuals in time T via a single integral over the density distribution function of the normal distribution, with the integral boundaries of $N_T - 0.5$ and $N_T + 0.5$.

For example, our tool enables us to calculate the whole distribution of surviving probabilities in 2007. Using the mortality rates tables published by the SSA, we can calculate the probability of individuals born in 1960 to survive until 2006 (included). The probability of individuals born in 1960 to survive until 2006 (included) is 0.89195. Thus, among every 1000 individuals born in 1960, 891.947 will survive till the end of 2006, on average. We can also tell that, on average, out of the 1000 original individuals, about 888 of them will survive till the end of 2007. But, there might be a given probability that the realized number of survivals will be much larger or much lower than the expected number. Thus, relying solely or mainly on the expected value of survivals, many institutions such as pension funds, geriatric institutions and the medical authorities in general, may end up extremely unbalanced. If such a company could estimate the distribution of the expected future number of deaths for each cohort, it could expand its forecasting tools and thus reduce its operational risks by using the confidence interval of the expected number of deaths instead of solely using the expected value.

We can also forecast the estimated distribution of future survivors in any period in the future. For example, we can forecast the distribution of survivors in 2015 for the same group of individuals born in 1960. Naturally, the SSA mortality tables do not go into the future. However, we can apply the current mortality rates data and use it as an estimate for the future mortality probabilities for the cohort born in 1960.

Continuing our example, the estimated probability for the cohort born in 1960 to survive until 2014 (included) is 0.85147 and thus the expected number of survivors in 2014 is 851.471 (out of the original 1000 individuals born in 1960). Using our technique and the forecasted survival probability for 2015 (0.99203), we can calculate the expected forecasted distribution of future survivors in 2015.

Thus, our paper contributes not only to the literature on the projection of mortality rates, but it also has significant practical implications because it enables the authorities, as well as other relevant institutions, to be better prepared for the upcoming future and to better handle unexpected changes associated with it.

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Appendix-A: Table-A.1. SSA data: mortality and survival rates for individuals born in 1960, for each year between 1960 and 2007

Year	Age at that year	Mortality rates	Survival rates
1960	0	0.02937	0.97063
1961	1	0.00174	0.99826
1962	2	0.00105	0.99895
.	.	.	.
.	.	.	.
2005	45	0.00375	0.99625
2006	46	0.00397	0.99603
2007	47	0.00421	0.99579

Appendix-B: The Mathematica code for the distribution of surviving individuals in 2007, for 1000 individuals born in 1960:

```
strm=OpenWrite ["…fill in the required output path…"]
num=1000
For[m=0,m<num,m++,Write[strm,Integrate[1/Sqrt[2*Pi*n*p*(1-p)]*Exp[-(x-n*p)^2/(2*n*p*(1-p))], {x, m-0.5, m+0.5},Assumptions→ {n=891.947,p=0.99579}]]]
```

Appendix-C: Table-C.1. The distribution of surviving individuals in 2007, for 1000 individuals born in 1960.

Number of surviving individuals	Probability
873	0.00000000000000
874	0.00000000000000
875	0.00000000002557
876	0.00000000071469
877	0.00000001534577
878	0.00000025319193
879	0.00000321078768
880	0.00003130266064
881	0.00023467355658
…	…
…	…
895	0.00047421435433
	Continue

896	0.00006996033194
897	0.00000793746239
898	0.00000069240502
899	0.00000004642806
900	0.00000000239241
901	0.00000000009471
902	0.00000000000000
903	0.00000000000000

Appendix-D: Table-D.1. Forecasted mortality and survival probabilities for the cohort born in 1960.

The cohort born at the year	**Age at the year 2007**	**Mortality rates**	**Survival rates**	**An estimation for the mortality and survival probabilities for the cohort born in 1960 for the year**
1959	48	0.004603	0.995397	2008
1958	49	0.005037	0.994963	2009
1957	50	0.005512	0.994488	2010
1956	51	0.006008	0.993992	2011
1955	52	0.006500	0.993500	2012
1954	53	0.006977	0.993023	2013
1953	54	0.007456	0.992544	2014
1952	55	0.007975	0.992025	2015