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REST BREAKS ARRANGE BASED ON EMPIRICAL STUDIES OF PRODUCTIVITY IN MANUFACTURING INDUSTRY

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ABSTRACT

This paper investigates how worker productivity differs in each consecutive work period based on tests conducted on a manufacturing company. We first employed a two-dimension fixed effect model to unbalanced panel data with 6,117,317 records which contain 102 employees and 1680 workdays. We also found that worker productivity between two adjacent rest breaks shows a U-shaped trend. This finding supplements the previous researches about interruption events, which only explored the effect of interruption after its occurrence. Based on our empirical findings, to obtain the lowest production system variability, a mathematical model which aims to optimally configure work and break scheduling of a production line is then presented. The optimal schedule is that one worker either begins their work when another worker from the same workstation starts their break immediately or has worked for a half of a completed work and break period. Our analysis combines empirical studies and model analysis to improve the operations management researches and practices.

Contribution/ Originality: This study is one of very few that finds the U-shaped trend between two adjacent rest breaks and builds a model to evaluate its economic value in manufacturing industry.

1. INTRODUCTION

The manufacturing system is often affected by various factors such as raw materials shortage, inventory, setup, maintenance, rework, buffer, machine failures and breaks which result in process variability. As a critical performance index, process variability plays an important role on production lines. In general, production variability increases the cost of operational processes and produces a negative effect on system performance. That production variability may cause increased process cost is widely recognized (Lee and Billington, 1993). The impact of variability in process management triangles (capacity utilization, variability and inventory) on productivity has been an important research topic in operations management (Klassen and Menor, 2007). Hopp and Spearman (1996) believe that increasing variability degrades the performance of a production system. Although many factors may incur variability in production processes, the assumption in previous researches takes people as fixed, unchanging, or exogenous entities (Lee and Tang, 1998; Roubos *et al.*, 2012). However, worker productivity can change due to psychological and physiological causes. Therefore, human behavior is also a source of production variability.

It is well known that modeling human behavior is a difficult task, so empirical research about human behavior becomes necessary. However, by analyzing 1,015 papers on the topic of operations management published in 2010 to 2015 in three top journals (*Management Science (MS*), *Manufacturing & Service Operations Management* (*M&SOM*) and *Production and Operations Management (POM*)), Ho *et al.* (2017) found only 18 percent of articles contained empirical analysis. The main reason, we think, is the lack of data. In the current manufacturing environment, many types of data can be generated during the operation processes of an enterprise, and more and more data are recorded by systems such as Manufacturing Execution System (MES), Enterprise Resource Planning (ERP) and Cyber-Physical Systems (CPS). The availability of production data, in turn, makes it possible to study the effect of human behavior. In recent years, there are some researches that study the productivity effect of humans. For example, some scholars study how interruption affects human productivity (Trougakos *et al.*, 2008; Pang and Whitt, 2009; Froehle and White, 2013; Kolbeinsson *et al.*, 2014; Lu *et al.*, 2014; Trougakos *et al.*, 2014; Sanderson and Grundgeiger, 2015; Hunter and Wu, 2016; Kim *et al.*, 2016; Andreasson *et al.*, 2017; Kolbeinsson *et al.*, 2017; Pasquale *et al.*, 2017; Cai *et al.*, 2018). Some scholars study how specialization and variety of tasks affect human productivity (Narayanan *et al.*, 2009; Kc and Terwiesch, 2011; Staats and Gino, 2012; Kc, 2013). Some scholars study how workload effects productivity (Kc and Terwiesch, 2009; Dietz, 2011; Tan and Netessine, 2014) and so on.

The truth is, the three factors (interruption, task variety and workload) may all affect worker productivity, individually or collectively, leading to variability in the production system. In this paper, we do not investigate the behavioral factors that affect productivity, but focus on worker performance variability between scheduled rest breaks. To the best of our knowledge, there have been few studies of this issue to date.

We study a production process of a manufacturing enterprise with traits of constant repetition, simple operation, single product and short processing time. Using a data-driven method described by Simchi-Levi (2014) unbalanced panel data with 6,117,317 records including 102 employees and 1680 workdays was analyzed. It was here that we found the U-shaped trend in worker productivity between two adjacent rest breaks.

The changes of productivity indicate the existence of productivity variability in the production processes. In previous studies, the researchers assumed that the production time of one item satisfies a certain distribution, usually an exponential or normal distribution because variability of production line is still hard to define (Whitt, 1995; Wu *et al.*, 2016). However, in this paper we found a specific distribution (U-shaped) of productivity with time between two adjacent rest breaks by using realistic data.

Variability is always the enemy of performance in a production system (Schonberger, 1982; Inman, 1993). Therefore, how to deal with productivity variability between two adjacent rest breaks is another critical problem. There are many researchers who have provided different methods to ease the stress of process variability. Gong *et al.* (2009) believed that the target of lean manufacturing is to exposure process variability which can then be reduced. Lee and Tang (1998) and Kapuscinski and Tayur (1999) reengineered the manufacturing process by reversing two consecutive stages. This process reversion reduced the variance and improved production performance. Li (2003) utilized simulation research and found that a workplace's performance will be improved by decreasing set-up and processing time variabilities.

In this study we developed a mathematical model to rearrange rest breaks that can improve process variabilities. We assumed that there were two identical workers in each workstation of a production line, hence, the two workers can share equal parts of the supplied materials and so be able to pass-on the same product to the next workstation. Here, we present a model designed to optimally configure the production and break scheduling with the lowest possible variability.

This paper is organized as follows. We execute an empirical analysis in part 2. In part 3, a system scheduling model with two identical workers in each workstation is formulated based on our empirical analysis. This part also compares the variability between two production style designs and discusses the optimal conditions of how to arrange work and break intervals. The paper concludes with a discussion of the management implications of our main findings.

2. EMPIRICAL ANALYSIS

In this part, data from a test conducted on a manufacturing company is collected in order to study worker productivity change between two adjacent rest breaks. Then, a two-dimensional fixed effect model is utilized for our analysis.

2.1. Data, Variables and Model

2.1.1. Data Collection and Processing

This study selects a production line's test of a manufacturing company during 2017.01.01 to 2017.12.31 as our dataset with traits of constant repetition, simple operation, single product and short test time. Worker's standard working hours are from 8:00 am to 20:00 pm. In practice, employees take rest every day at relatively fixed times which are morning rest (10:00 am to 10:15 am), lunch break (12:00 pm to 13:00 pm), afternoon rest (15:00 pm to 15:15 pm) and dinner break (17:00 pm to 18:00 pm). According to responses from firm's managers and front-line employees, the actual rest and meal times are slightly longer than scheduled. Excluding a very small number of recording errors and missing data, we selected an unbalanced panel data with 6,117,317 records, 102 employees and 1680 workdays. Among them, the operator with highest attendance rate worked 110 days while the lowest worked only one day. Figure A1 in appendix A indicates its density distribution. Therefore, each operator works about 16 days on average in this year.

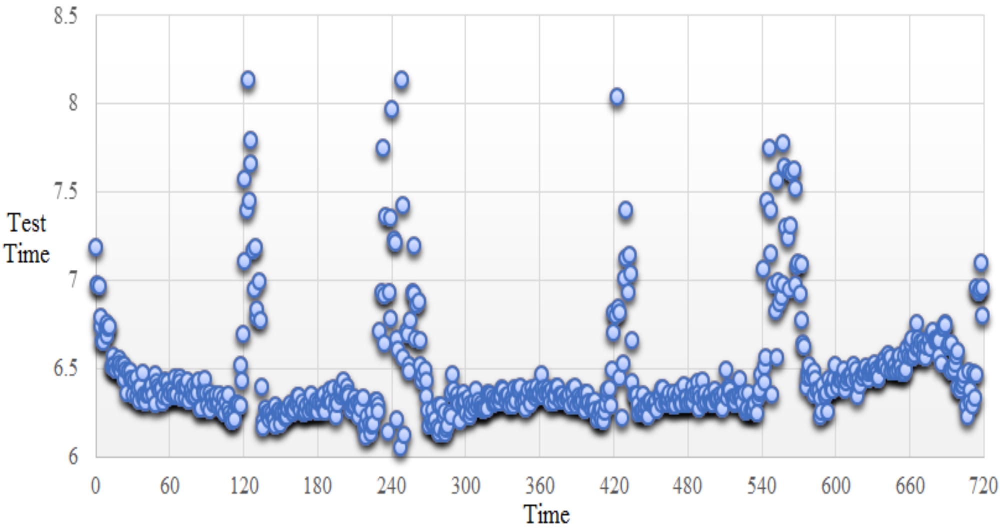

Figure-1. Average test time per minute of during 8:10 am-20:00 pm.
Source: Developed by authors according to data sources from the selected manufacturing company.

In order to describe the productivity change roughly, a one minute interval is adopted to capture each worker's average test time. Figure 1 shows the average test time calculated by one minute of all operators during the work day. The horizontal axis represents the number of one minute interval, from one to 720. The number "1" is the time interval 8:00 am-8:01 am and "720" represents the time interval 19:59 pm-20:00 pm. In Figure 1, the average test time during or near the break interruption is apparently larger than in other periods. There are two reasons for this: (1) number of actual production tests around break interruption are less than non-interrupted periods; and (2) the test time has a larger fluctuation either before or after the interruption because of setup time, work delay or other intangible factors.

The dayshift is divided into five consecutive stages of work based on break interruptions as shown in Figure 1. As indicated, the standard rest time is 15 minutes, however, in practice, employees' rest time always fluctuates around 15 minutes, especially for the short period following rest breaks. This article selects the period 10:00 am to10:20 am and 15:00 pm to 15:20 pm as the morning rest and afternoon rest respectively. For meal breaks, the tested enterprise adopted a meal-break strategy in which workers have lunch in batches to reduce crowding in cafeteria during the Lunch Break. The three batches for lunch are 11:40 am to12:40 pm, 11:50 am to 12:50 pm and 12:00 pm to13:00 pm, respectively. Generally, the lunch break starts between 11:40 am and 12:00 am. Therefore, the period 11:50 am to 13:00 pm is taken as the lunch break in this study. Conversely, for the dinner break, batches arrangement is not adopted because the demand at dinner time is relatively light, and the scheduled time is between 17:00 pm and 18:00 pm. Frequency of other non-rest and non-dining interruptions are very low in this company. Therefore, it is feasible to define the interruption in a practical situation. In addition, the timeframe of rest breaks and the five stages are shown in Table 1. From Figure 1, it is obvious that the starting and ending of the dayshift causes only a small fluctuation in productivity. We excluded the first and last 10 minutes' data from every dayshift.

Table-1. The definition of interruption and working stages.

Periods	Time	Periods	Time
Stage1	8:10-10:00	Afternoon rest	15:00-15:20
Morning rest	10:00-10:20	Stage4	15:20-17:00
Stage2	10:20-11:50	Dinner break	17:00-18:00
Lunch break	11:50-13:00	Stage5	18:00-19:50
Stage3	13:00-15:00		

Notes: The division of scheduled break interruptions and working stages results from the production plan of selected company.

In practice, the test time of this procedure is approximately 7 seconds and some abnormal interruptions may affect our analysis. Based on average test time of one every minute, this paper deals with the original data in the following steps. Firstly, we find the rest break location of each operator's workday according to Table 1 which includes all the test data with test time of less than 420s between 8:10 am-19:50 pm. Secondly, the mean and standard deviation of average test time are calculated in each stage. σ principle is employed to remove some abnormal test records. Because our test time is positive, we only consider the upper bound of σ principle. 1.5σ is adopted and we obtain the mean (μ=10.18s) and upper bound ($\mu+1.5\sigma$=34.72s) by adding up all five stages. We exclude test records with average test times larger than 34.72s and the detail is shown in Table A1. Because the number of total records are large enough and the test records with test time higher than 34.72s occupy only a very small proportion in our data, thus regardless of whether σ is 1.5σ or 3σ it would have little impact on our analysis and so our approach is reasonable.

2.1.2. Variables

2.1.2.1. Dependent Variable

In this study, we directly take the average test time as the measurement of worker productivity. Our empirical study takes one minute as the time interval. The productivity is calculated as $productivity = Test\ number / Actual\ test\ time$. $Test\ number$ denotes the quantity of test production within one minute, and $Actual\ test\ time$ represents the difference between end time of the last production and start time of the first production during one minute intervals, which is always equal to or less than 60s.

2.1.2.2. Independent Variable

The independent variable in this paper is time. To study worker productivity change over time, the following definition is adopted. The meaning of our independent variable $Time_{id}^{n}$ is similar to that of the "Elapsed working hours" of Lu *et al.* (2014). Its size represents successive work time stages which measures the time length from the beginning of each work stage per minute. Superscript n represents different stages and its range is $[1,2,3,4,5]$. Subscript i indicates worker and d indicates day. Taking stage 2 as an example, this stage locates at 10:20 am to 11:50 am and has 90 minutes in total, and we define the value of $Time_{id}^{2}$ with one minute interval located on $[1,2,...,90]$. Thus, $Time_{id}^{2}$ equals to "1" which represents the time of test records as between 10:20 am and 10:21, and $Time_{id}^{2}$ equal 90 which means the time of test records is between 11:49 am and 11:50 am. In order to explore the nonlinear relationship between worker productivity and elapsed time from the beginning of each stage, the quadratic term $\left(Time_{id}^{n}\right)^{2}$ is also employed.

2.1.3. Model

To analyses our unbalanced panel data and to control confounders, a two-dimension fixed effect model which is able to capture the difference across the workers and dates, is implemented in this part because different workers may have inconsistently average productivity levels. Our model explores the productivity trend of each five stages.

According to our unbalanced panel data, the average test time $productivity_{id}$ is taken as the dependent variable which measures the worker productivity per minute. The longer the test time, the lower the productivity is. On top of that, two fixed effects are captured to control for the confounders. The worker fixed effect, denoted as λ_i, controls for the difference among workers, which the heterogeneity of date is controlled by date fixed effect λ_d. In this paper, the worker level is utilized for cluster standard errors to control for any other serial correlation and heterogeneities (Bertrand *et al.*, 2004; Cai *et al.*, 2018). Research from Cai *et al.* (2018) cluster standard errors at machine level, but, there are only two machines in our study. Those two machines which have almost the same condition would not affect worker productivity, therefore, we select the cluster standard errors at worker level. One minute interval is selected in each work stage. Model (1) is implemented to describe the detail of the nth stage $(n=1,2,3,4,5)$. Just like the following Equation 1 in each stage, the dependent variable is still the average test time $productivity_{id}^{n}$ of one minute while the independent variable contains linear term $Time_{id}^{n}$ and quadratic item $\left(Time_{id}^{n}\right)^{2}$ for exploring the nonlinear relationship between productivity and time. γ^{n} and θ^{n} are coefficients of $Time_{id}^{n}$ and $\left(Time_{id}^{n}\right)^{2}$ respectively.

$$productivity_{id}^{n}=\gamma^{n}\times Time_{id}^{n}+\theta^{n}\times\left(Time_{id}^{n}\right)^{2}+\lambda_i+\lambda_d+\varepsilon_{id}^{n} \quad (1)$$

Our models are all carried out by the code "reg2hdfe" from Stata which is utilized in the research conducted by Cai *et al.* (2018).

2.2. Empirical Findings

This subpart presents the empirical findings of our model based on test data. We implement model (1) and obtain the results for each stage in Table 2 with two-dimension fixed effect model.

Table 2 reports the results of model (1), with each column reporting the results for a corresponding stage. It is apparent that the coefficients of stage1-5 are all statistic significant at 5% level, and the estimation of the coefficients indicates that there is a quadratic function relationship between worker productivity and successive work time between two adjacent rest breaks. Productivity of stage 1 has a reverse trend compared with other four stages. In stage 1, the work time is 90 minutes in total while the vertex of the quadratic function is located at $Time_{id}^{1}=145$, therefor, worker productivity is going to increase monotonically.

Table-2. Regression conclusions of model (1).

Variable	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
$Time_{id}^{n}$	-0.007521***	0.005700***	0.004138***	0.003155***	0.009672***
	(0.001)	(0.001)	(0.001)	(0.001)	(0.002)
$\left(Time_{id}^{n}\right)^2$	0.000026**	-0.000070***	-0.000036***	-0.000031***	-0.000062***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Controls					
Worker FE	Yes	Yes	Yes	Yes	Yes
Date FE	Yes	Yes	Yes	Yes	Yes
Observations	135,551	114,079	154,064	124,447	106,223
R-squared	0.276	0.394	0.35	0.34	0.15

Notes: All the regression's dependent variable is test time index $productivity_{id}^{n}$ and the independent variable is time variable with 1 minute interval $Time_{id}^{n}$ and its quadratic item $\left(Time_{id}^{n}\right)^2$, $(n=1,2,3,4,5)$. Worker level is clustered to calculate the standard errors. Robust standard errors in brackets *** p<0.01, ** p<0.05, * p<0.1.

This inconsistence between stage 1 and stage 2-5 may arise from many aspects. For example, after an overnight rest, it is of little possibility for employees to concentrate on their work at the very beginning of the second day. In addition, equipment preparation and worker proficiency accumulation are both time-consuming. Therefore, worker productivity shows a decreasing trend over time in this stage. On the contrary, all of the stage 2-5 have a negative coefficient of $\left(Time_{id}^{n}\right)^2$ but a positive for $Time_{id}^{n}$. According to the estimated coefficients, the largest value of $productivity_{id}^{n}$ is obtained approximately at the 40th, 57th, 51th and 78th minutes in stage 2-5, respectively. Additionally, in stage 2-5, the dependent variable $productivity_{id}^{n}$ follows an inverted U-shaped function for $Time_{id}^{n}$, indicating a U-shaped trend of worker productivity will decreases first and then increases during each of stage 2-5. Considering stage 2-4 are all between two adjacent rest breaks, our empirical finding supports that worker productivity displays a U-shaped trend between two adjacent rest breaks.

2.3. Robustness Checks

To test the robustness of our analysis results of U-shaped productivity trend in Table 2, following checks are made. Firstly, we focus on the dependent variable. Our normality tests report that the average test time per minute disobeys a normal distribution. Therefore, the natural log of $productivity$ is taken as new dependent variable, and the regression results are presented in Table 3. It is noted that the coefficients are basically consistent and significant as with previous findings in Table 2. The sign of all coefficients are also the same as those in Table 2, along with slight differences in magnitude. This test confirms that the form of the dependent variable has little effect on our earlier conclusions, demonstrating the robustness of our empirical findings.

Secondly, we extend the effect of unobservable factors on productivity by enlarging test records with longer test times. According to the investigation of front-line employees, the rest time in the morning or afternoon is always longer than half of the fixed rest time (15 minutes). We took test records with the test time and the outcomes fell into the range of 34s – 420s. As in Table 2, we took the test times which were less than 420s as the new dependent variable. Table 4 shows that the other four regressions have the same coefficients sign as those in Table 2, except for stage five. We found that only the coefficient of the quadratic item in stage five is significant at a level of five percent. Then, a linear regression is employed and the coefficient (0.039922) is positively significant at one percent. Two reasons may account for the difference between stage five and stages two to four. At first, more interference factors may appear when employees are about to finish their day's work. In addition, there are relatively fewer test records and larger variances in test times at the end of the work day. Therefore, abnormal test records are more likely to appear at that time. This means that our earlier analysis is still robust even though we enlarged the scope of dependent variable.

Table-3. Regression conclusions of model with $\ln(productivity)$.

Variable	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
$Time_{id}^n$	-0.000866***	0.000842***	0.000483***	0.000339**	0.000848***
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
$(Time_{id}^n)^2$	0.000002**	-0.000010***	-0.000005***	-0.000003**	-0.000004**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Controls					
Worker FE	Yes	Yes	Yes	Yes	Yes
Date FE	Yes	Yes	Yes	Yes	Yes
Observations	135473	114021	153988	124312	105752
R-squared	0.222	0.314	0.273	0.27	0.136

Notes: All the regression's dependent variable is test time index $\ln(productivity_{id}^n)$ and the independent variable is time variable with 1 minute interval $Time_{id}^n$ and its quadratic item $(Time_{id}^n)^2$, $(n=1,2,3,4,5)$. Worker level is clustered to calculate the standard errors. Robust standard errors in brackets *** p<0.01, ** p<0.05, * p<0.1.

Table-4. Regression conclusions of model with $productivity < 420s$.

Variable	Stage 1	Stage 2	Stage 3	Stage 4	Stage 5
$Time_{id}^n$	-0.034539***	0.016929***	0.021935***	0.028921***	0.008659
	(0.006)	(0.006)	(0.005)	(0.007)	(0.012)
$(Time_{id}^n)^2$	0.000185***	-0.000287***	-0.000179***	-0.000247***	0.000313**
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Controls					
Worker FE	Yes	Yes	Yes	Yes	Yes
Date FE	Yes	Yes	Yes	Yes	Yes
Observations	137269	115386	155879	126008	108069
R-squared	0.079	0.107	0.088	0.086	0.035

Notes: All the regression's dependent variable is test time index $\ln(productivity_{id}^n)$ which is less than 420s and the independent variable is time variable with 1 minute interval $Time_{id}^n$ and its quadratic item $(Time_{id}^n)^2$, $(n=1,2,3,4,5)$. Worker level is clustered to calculate the standard errors. Robust standard errors in brackets *** p<0.01, ** p<0.05, * p<0.1.

Thirdly, we divided each of stages two to five into two sides based on the lowest vertex of the U-shaped trend. Coefficients of Table 2 display that the timing for obtaining the worst productivity locates on the 40^{th}, 57^{th}, 51^{st} and 78^{th} minutes from the beginning of stages two to five. We took these time points as the boundary value and applied two linear models for each stage. The upper half of Table 5 demonstrates the regression results of the left side, while the bottom half shows the regression results of right side in stages two to five. We found that coefficients of left side are all positive even though stages two and four are not significant at a level of 10 percent, and the coefficients of the right side are all negative and significant at 10 percent. This is consistent with U-shaped productivity finding in Table 2.

Table-5. Two sides' regression of U-shaped in stage 2-5.

Stage	Stage 2	Stage 3	Stage 4	Stage 5
Time interval	1-40 minutes	1-57 minutes	1-51 minutes	1-78 minutes
$Time_{id}^{n}$	0.001175	0.001356*	0.001051	0.004852***
	(0.001)	(0.001)	(0.001)	(0.001)
Observations	51670	74239	65871	86305
R-squared	0.399	0.35	0.366	0.175
Time interval	41-90 minutes	58-120 minutes	52-100 minutes	79-110 minutes
$Time_{id}^{n}$	-0.003634***	-0.002739***	-0.001955*	-0.031809***
	(0.001)	(0.001)	(0.001)	(0.003)
Observations	62409	79825	58576	19918
R-squared	0.399	0.357	0.324	0.115

Notes: All the regression's dependent variable is test time index $\ln\left(productivity_{id}^{n}\right)$ and the independent variable is time variable with 1 minute interval $Time_{id}^{n}$, $\left(n=1,2,3,4,5\right)$. Worker λ_i and day λ_d fixed effect are adopted and worker level is clustered to calculate the standard errors. Robust standard errors in brackets *** p<0.01, ** p<0.05, * p<0.1.

3. REST SCHEDULING BASED ON "U-SHAPED" PRODUCTIVITY

3.1. Production Setting and Model Assumptions

In many manufacturing companies process variability is a critical factor which deteriorates the system's performance (Hopp and Spearman, 1996). How to arrange the production line to mitigate the effect of variability is an important consideration. In the company we investigated, there are two workers in each workstation of a production line where they share the supply of parts and materials and pass-on the same product to the next workstation as described by Figure 2. The circles with solid line represent the workers located on both sides of the production line, while circles with dotted line represent the workstations on a conveyor belt. The parts material flow step-by-step from workstation one to workstation m. Combining our empirical findings and manufacturing practice, a model is thereby built up to determine the optimal work and break plan, and to research the lowest variability of the production line.

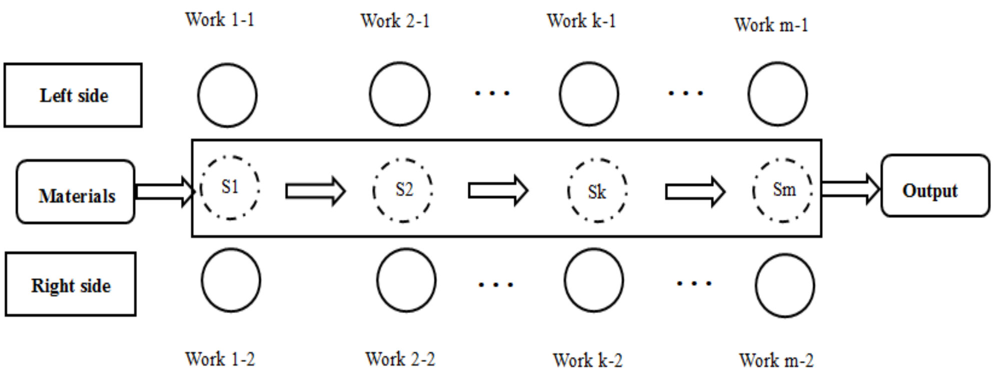

Figure-2. Two production line with joint parts material supply.
Source: Developed by authors according to above empirical conclusion.

In order to abstract and simplify our model, we made several assumptions for our realistic production environment. From Figure 2, we defined several parameters with detailed a description in Table 6. Then, we integrated the empirical U-shaped trend of worker productivity between two adjacent rest breaks upon which to base the following assumptions.

Table-6. Description of model parameters.

Parameters	Description
s	Consecutive work time between two adjacent rest breaks.
T_b	Worker fixed break time.
T_i	Break interval time between two workers from the same workstation.
t	Elapsed time since the worker begins his/her work after the last break interruption.
p	The rate of break time and consecutive work time: T_b / s.
E	Worker productivity during a consecutive work time: $[0, s]$.
a, b and c	Parameters of quadratic function with U-shaped productivity trend.

Source: Developed by authors according to our theory model.

3.1.1. Assumptions

1. Two identical operators work in each workstation of production line. The two workers share the supply of parts and materials and pass-on the same product to the next workstation. The scheduled break time is T_b and the successive work time between two adjacent break interruptions is denoted as s.
2. In each successive work period $[0, s]$, workers have the same U-shaped productivity trend and quadratic equation $E = at^2 - bt + c$ over time, E represents worker productivity, t represents elapsed time, and a, b and c are all positive parameters. Because our productivity is positive, the inequality $b^2 < 4ac$ is satisfied. We assume the U-shaped productivity trend is symmetrical which gains the maximum productivity in the middle of a consecutive work time, hence, $s = b/a$.
3. Two different production style designs are considered in Figure 2. The first style is mainly based on current practice which two workers at each workstation have the same work and break rhythms. Keeping the total work and break time constant, the second style arranges a worker of one workstation to begin their work and break with a fixed interval time T_i compared to another one.
4. We assume T_b and T_i are both less than s and $T_b < T_i$. Additionally, the rate of break time and successive work time is p ($p \in (0,1)$, $p = \frac{T_b}{s}$).

Based on above four assumptions, we can describe parameters in Table 6 and find Equation 2 and Equation 3 as follows. Those two equations show the relationships among parameters in Table 6.

$$0 < T_b = p\frac{b}{a} \le T_i \le s = \frac{b}{a},\ p \in (0,1) \qquad (2)$$

$$b^2 < 4ac \quad (3)$$

3.2. Variability of Two Different Production Styles

This subsection discusses two production style designs based on Figure 2 with the same productivity trend but different production and break plans.

Production style 1: Same work and break rhythms for workers of each workstation.

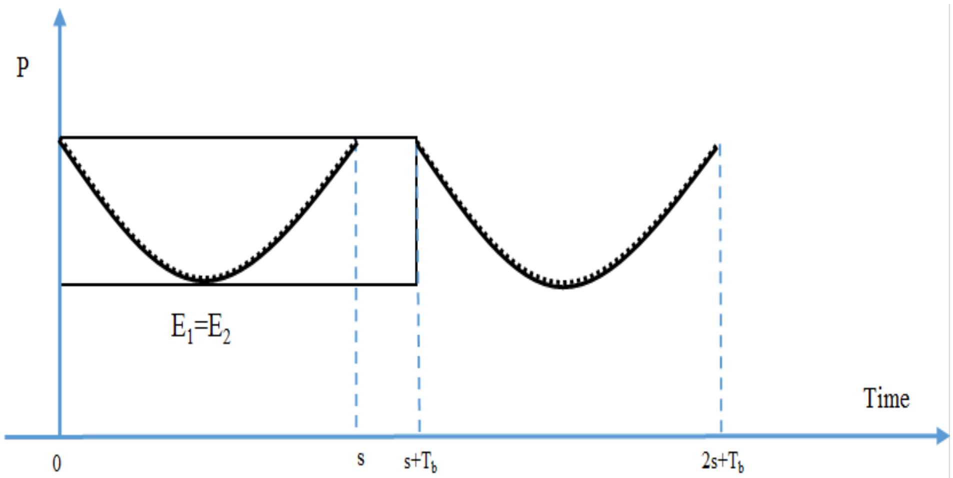

Figure-3. Same work and break rhythms of each workstation.
Source: Developed by authors according to different condition of parameters in our model.

Figure 3 depicts the current manufacturing setting. The solid and dashed lines represent productivity change of two workers from each side of a workstation, and those two workers have same work and break rhythms. The horizontal axis is elapsed time calculated from the beginning of the work and the vertical axis represents worker productivity. One completed cycle period ($s+T_b$) is considered, as shown in rectangle of Figure 3. In this circumstance, each of the two workers from one workstation satisfy the same productivity change $E_1 = E_2 = at^2 - bt + c$ during the period $t \in [0, s]$ and there are no production activities in their joint break time $[s, s+T_b]$. In this paper, we mainly concentrated on the productivity variability of workers and considered the productivity variance of an entire production line as our variability. This is consistent with He *et al.* (2007) who took variance of test time as the variability. Thus, mean (E_{Same}) and variance (Var_{Same}) of production style one can be obtained as the following Equation 4 and Equation 5.

$$E_{Same} = \frac{1}{s+T_b}\int_0^{s+T_b} 2(E_1+E_2)dt = \frac{1}{s+T_b}\int_0^{s} 2(E_1+E_2)dt \quad (4)$$

$$\begin{aligned} Var_{Same} &= \int_0^{s+T_b} \frac{1}{s+T_b}\left((E_1+E_2)-E_{Same}\right)^2 dt \\ &= \frac{1}{s+T_b}\left(\int_0^{s}\left(2(E_1+E_2)-E_{Same}\right)^2 dt + \int_s^{s+T_b}\left(0-E_{Same}\right)^2 dt\right) \end{aligned} \quad (5)$$

Production style 2: Different work and break rhythms with interval T_i for each worker of a workstation.

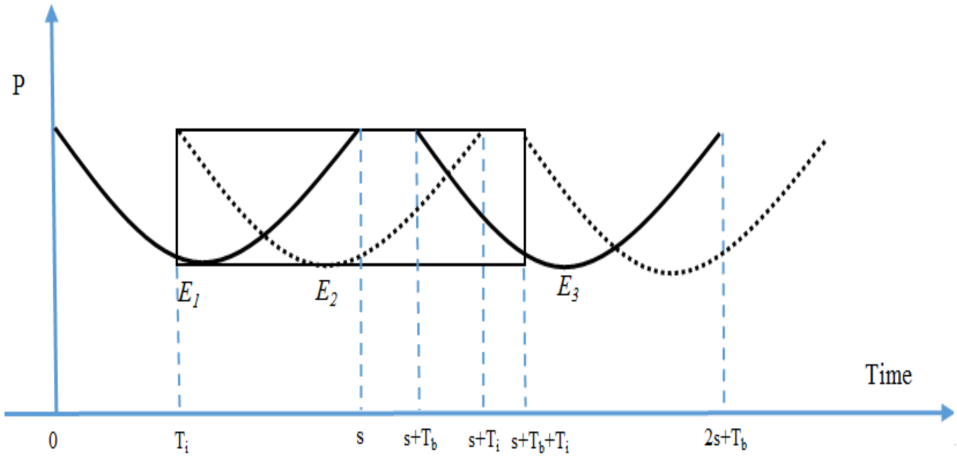

Figure-4. Different work and break rhythms of each workstation.
Source: Developed by authors according to different condition of parameters in our model.

Similar to what was presented in Figure 3, one cycle period ($s+T_b$) with same productivity trend, as shown in rectangle of Figure 4 is selected. There is a work and break interval between workers from the same workstation which means one worker always starts their work and break T_i lagged than another one. During the selected period $[T_i, s+T_i+T_b]$, three of the same productivity trends with different function forms (E_1 and E_3 belong to one worker and E_2 belongs to another worker from the same workstation) are gained overtime:

$E_1 = at^2 - bt + c, t \in [T_i, s]$, $E_2 = a(t-T_i)^2 - b(t-T_i) + c, t \in [T_i, s+T_i]$ and $E_3 = a(t-s-T_b)^2 - b(t-s-T_b) + c$, $t \in [s+T_b, s+T_b+T_i]$. Therefore, mean ($E_{Interval}$) and variance ($Var_{Interval}$) of production style 2 can be obtained as the following Equation 6 and Equation 7.

$$E_{Interval} = \int_{T_i}^{s+T_b+T_i} \frac{1}{s+T_b}(E_1+E_2+E_3)dt = \frac{1}{s+T_b}\left(\int_{T_i}^{s} E_1 dt + \int_{T_i}^{s+T_i} E_2 dt + \int_{s+T_b}^{s+T_b+T_i} E_3 dt\right) \quad (6)$$

$$Var_{Interval} = \int_{T_i}^{s+T_b+T_i} \frac{1}{s+T_b}\left(E_1+E_2+E_3-\overline{E_{1,2,3}}\right)^2 dt$$
$$= \frac{1}{s+T_b}\left(\begin{array}{l}\int_{T_i}^{s}\left(E_1+E_2-\overline{E_{1,2,3}}\right)^2 dt + \int_{s}^{s+T_b}\left(E_2-\overline{E_{1,2,3}}\right)^2 dt \\ + \int_{s+T_b}^{s+T_i}\left(E_2+E_3-\overline{E_{1,2,3}}\right)^2 dt + \int_{s+T_i}^{s+T_i+T_b}\left(E_3-\overline{E_{1,2,3}}\right)^2 dt\end{array}\right) \quad (7)$$

In this paper, the unique difference between production styles one and two is the starting work and break interval time T_i. It is realized that, for the two workers from the same workstation, at least one of them would work continuously for the entire time period. In other words, each workstation must keep running during a complete shift. This arrangement not only mitigates the crowding of materials supply, but also makes the workstation productivity more stable than that in production style one, with the same work and break rhythms. It is apparent that the average productivity of those two production style designs are equivalent. Does variability make a difference during the successive work and break time ($T_b + s$)? The following theorem one compares workstation variability difference measured by variance of worker productivity between production style one and production style two.

Theorem 1: In a successive work and break period $T_b + s$, *workstation of production style 2 has the same productivity mean but lower variability* compared *with production style 1:* $E_{Interval} = E_{Same}$ *and* $Var_{Interval} < Var_{Same}$.

In terms of keeping the output of a production line constant, theorem one demonstrates that a workstation of production style two experiences a lower process variability. Keeping the production line running provides a useful guidance to firms that enables managers to reduce a production line's variability by rearranging the staggered break times. In practice, this finding may provide a simple and feasible method for firms to utilize the U-shaped trend of workers' productivity sufficiently to reduce their production variability.

As for production style two with different rhythms, we take the interval time T_i between two workers from the same workstation as the decision variable to explore how the variability of a production line fluctuates. According to Equations 5 and 6, we can obtain the following theorem:

Theorem 2: In a successive work and break period $T_b + s$ *with work and break interval* $T_i \in [T_b, s]$, *the variance of workstation* $Var_{Interval}$ *has* only *one extremum at* $T_i = \frac{1}{2}(T_b + s)$ *when parameters satisfy the condition* $0 < b^2 \le 24acp/(3p+1-p^3-3p^2)$ *; there are three extremums at* $T_i = \frac{1}{2}(T_b + s) \pm \sqrt{\left[(3p+1-p^3-3p^2)b^2 - 24acp\right] / \left[4a^2(p+1)\right]}$ *or* $T_i = \frac{1}{2}(T_b + s)$ *when parameters satisfy the condition* $24acp/(3p+1-p^3-3p^2) < b^2 \le 4ac$.

Theorem 2 shows the relationship between workstation's variability $Var_{Interval}$ with different rhythms and interval T_i in production style two. Our calculations show that $Var_{Interval}$ is a fourth-order polynomial function in T_i, and the interval time T_i has a relatively complicated effect on $Var_{Interval}$. In our special context $T_i \in [T_b, s]$, there are either one extremum or three extremums, and $Var_{Interval}$ has a symmetrical property about the midpoint of $T_b + s$. Based on the conclusions obtained in theorem two, the optimal conditions of the minimum variability that the production line gains are further explored in theorem three.

Theorem 3: In a successive work and break period $T_b + s$ *with work and break interval* $T_i \in [T_b, s]$, $Var_{Interval}$ *gets its the minimum at* $T_i = \frac{1}{2}(T_b + s)$ *when*

$$3p^3 + 5p^2 + 5p - 1 \le 0 \text{ and}$$

$$48acp/\left(-3p^3 - 5p^2 + 7p + 1\right) < b^2 \le 24acp/\left(2p\left(2 - p^2 - p\right)\right);$$

$Var_{Interval}$ *gets the minimum at* $T_i = T_b$ *or* $T_i = s$ *when*

$$3p^3 + 5p^2 + 5p - 1 > 0 \text{ or } 3p^3 + 5p^2 + 5p - 1 \le 0 \text{ and}$$

$$24acp/\left(3p + 1 - p^3 - 3p^2\right) < b^2 < 48acp/\left(-3p^3 - 5p^2 + 7p + 1\right) \text{ or}$$

$$0 < b^2 \le 24acp/\left(-p^3 - 3p^2 + p + 1\right).$$

Theorem 3 shows the optimal schedule of work and break which gains the lowest workstation variability. Two optimal work and break arrangements appear when the work and break interval are $T_i = \frac{1}{2}\left(T_b + s\right)$ and $T_i = T_b \text{ or } s$, respectively. Figure 5 displays those two cases which have the lowest variability of our workstation. The left subfigure represents the case that minimum variability is obtained when the starting work and break interval time is half of a completed cycle period (break time plus a consecutive work time: $T_b + s$). The right subfigure shows another case with lowest variability when the starting work and break interval is same as the break time T_b or a consecutive work time s. The practice implication of Figure 5 suggests that the manager should layout their production line design with the following methods. In order their work when another worker from the same workstation starts their break immediately, or has worked for a half of a completed work and break period. This conclusion is very simple and feasible to implement in practice, and is able to help manufacturing companies reduce their production variability.

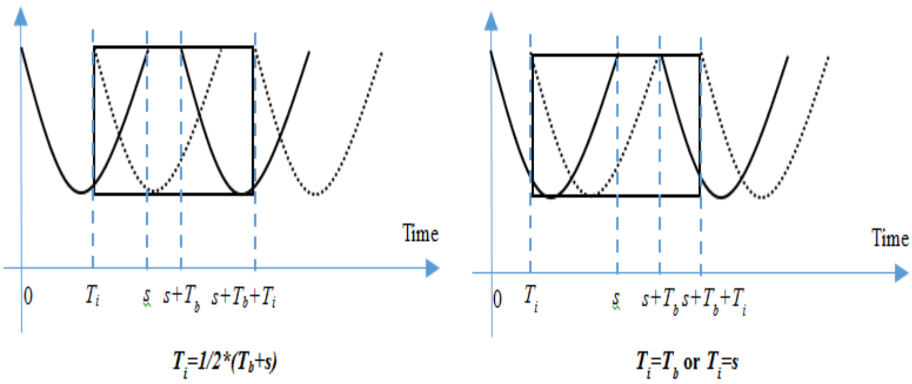

Figure-5. Two optimal designs with lowest variability of production style 2.
Source: Developed by authors according to different condition of parameters in our model.

4. CONCLUSION

In this paper, we explored two critical questions. Firstly, utilizing a production test procedure with constant repetitions, simple operation, single product, and short test time from a manufacturing enterprise, the productivity trend between two adjacent rest breaks was analyzed by employing the two-dimension fixed effect model. Empirical findings show that the trend of productivity between two adjacent break interruptions has a U-shaped form. To best of our knowledge, it is the first time that a U-shaped trend of worker productivity during successive work time between two adjacent rest breaks has been discovered within a selected manufacturing environment. Secondly, based on the findings from empirical research, we concentrated on a special production line application where each workstation has two identical workers. The two workers have the same "U-shaped" productivity trend and each worker has totally equal work and break time. Dividing the system into two production style designs by a work and break interval between two workers of a same workstation, a mathematical model was constructed to compare the variability of those two production style designs, and to optimally configure the production and break scheduling with lowest variability. Additionally, theorems one to three rising from model analysis demonstrate that production style with a work and break interval will have smaller a variability than a production style design with the same work and break rhythms. To obtain the lowest variability, it is optimal for manufacturers to arrange their production line as follows: one worker either begins their work when another worker from the same workstation starts their break immediately, or has worked for a half of a completed work and break period.

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APPENDIX

Appendix A

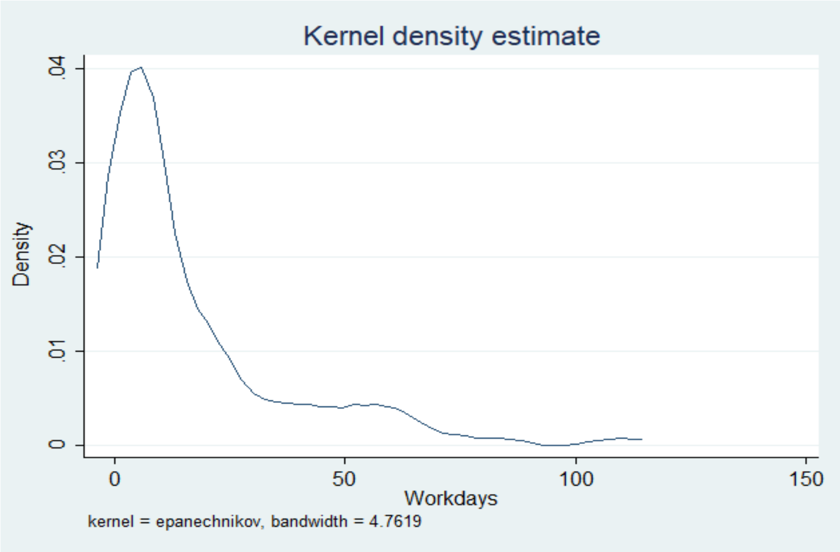

Figure-A1. Density distribution of working days.
Source: Developed by authors according to data sources from the selected manufacturing company.

Table-A1. The mean and standard deviation from data cleaning under 1.5 σ principle.

Stages	Mean	Std	Mean+1.5std
Stage 1	9.805785276	14.90138151	32.15785753
Stage 2	9.601312061	15.53169801	32.89885907
Stage 3	9.877705694	15.84673547	33.6478089
Stage 4	10.06244439	16.5711442	34.91916069
Stage 5	11.5406807	18.94914722	39.96440153
Mean	10.17758562	16.36002128	34.71761755

Source: Developed by authors according to data sources from the selected manufacturing company.

Appendix B

Proof of Theorem 1

In view of the difference of workstation variability between production style 1 and 2, our proof shows that all of the maximum point of $Var_{Interval}$ are less than the value of Var_{Same}. Two cases in our model are described in theorem 2 and 3 and the corresponding Figure A2 and Figure A3 are utilized to describe the Case 1 and Case 2, respectively.

Case 1 Figure A2: When $0 < b^2 \le \dfrac{24acp}{3p+1-p^3-3p^2}$, $Var_{Interval}$ is a concave function in T_i and its maximum can be obtained at $T_i = \dfrac{1}{2}\left(\dfrac{b}{a}+T_b\right)$.

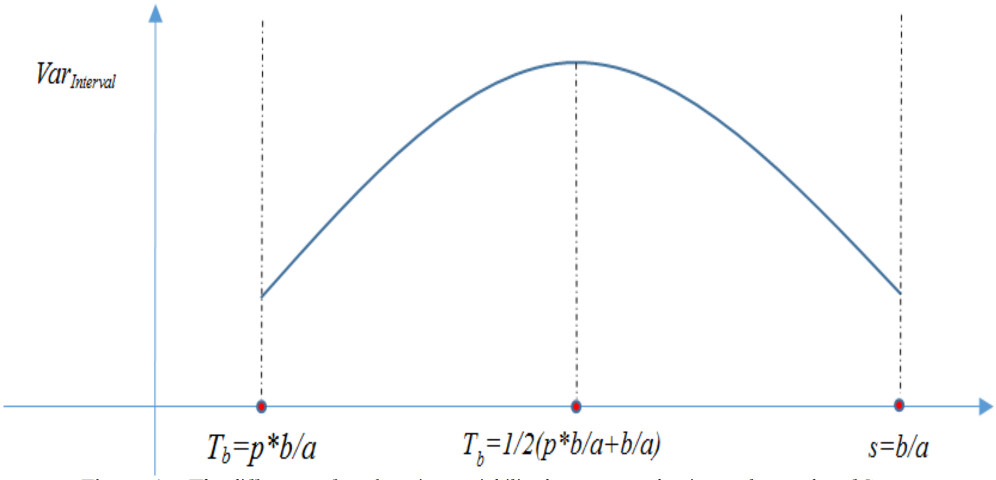

Figure-A2. The difference of workstation variability between production style 1 and 2 of Case 1.
Source: Developed by authors according to our theory model.

$$Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}} - Var_{Same}$$
$$= \frac{-\left(480a^2c^2p + 80ab^2cp^3 - 240ab^2cp + b^4p^5 + 5b^4p^4 - 10b^4p^3 - 10b^4p^2 + 25b^4p + 5b^4\right)}{240a^2(1+p)}$$

We define the numerator as a quadratic function $w\left(b^2\right)$ of b^2.

$$w\left(b^2\right) = 480a^2c^2p + 80ab^2cp^3 - 240ab^2cp + b^4p^5 + 5b^4p^4 - 10b^4p^3 - 10b^4p^2 + 25b^4p + 5b^4$$
$$= \left(p^5 + 5p^4 - 10p^3 - 10p^2 + 25p + 5\right)\left(b^2\right)^2 + \left(80acp^3 - 240acp\right)b^2 + 480a^2c^2p$$

$$\Delta = \left(80acp^3 - 240acp\right)^2 - 4\left(p^5 + 5p^4 - 10p^3 - 10p^2 + 25p + 5\right)480a^2c^2p$$
$$= 640a^2c^2\left(7p^6 - 15p^5 - 30p^4 + 30p^3 + 15p^2 - 15p\right)$$

Let $z(p) = 7p^6 - 15p^5 - 30p^4 + 30p^3 + 15p^2 - 15p$ and we find that $z(p)$ is either less than or equal to zero when $p \in [0,1]$. In addition, the quadratic item coefficient of $z(p)$ is $p^5 + 5p^4 - 10p^3 - 10p^2 + 25p + 5$ is positive under $p \in [0,1]$. Thus, we can get that $w\left(b^2\right) \square\ 0$ and $Var_{Interval} < Var_{Same}$ in case 1.

Case 2 Figure A3: When $\frac{24acp}{3p+1-p^3-3p^2} < b^2 \le 4ac$, $Var_{Interval}$ is a concave function in T_i and get its maximum at $T_i = \frac{1}{2}(T_b + s) \pm \sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2 - 24acp}{4a^2(p+1)}}$.

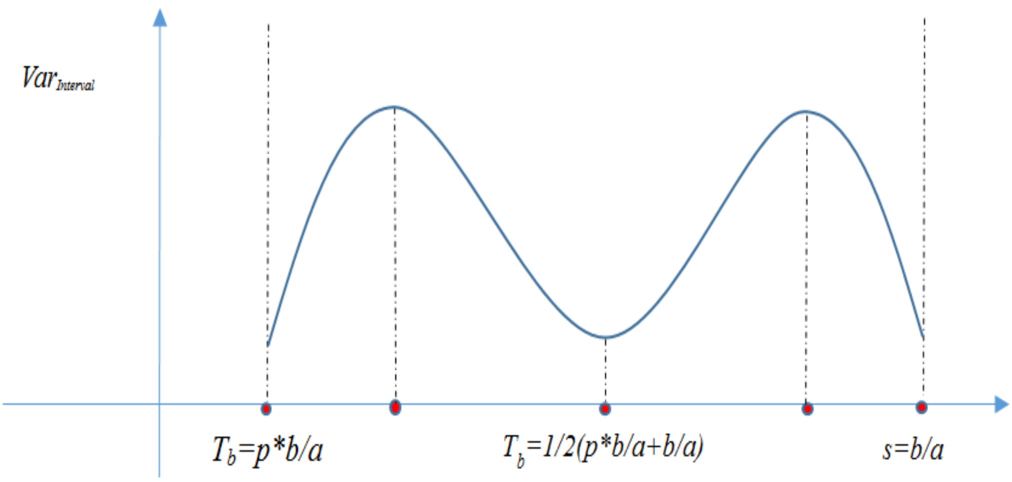

Figure-A3. The difference of workstation variability between production style 1 and 2 of Case 2.
Source: Developed by authors according to our theory model.

$$Var_{Interval}\Big|_{T_i=\frac{1}{2}(1+p)\frac{b}{a}\pm\sqrt{\frac{(3p+1-p^3-3p^2)b^2-24acp}{4a^2(p+1)}}} - Var_{Same}$$

$$= \frac{p\begin{pmatrix}600a^2c^2p-120a^2c^2+40ab^2cp^3+160ab^2cp^2-120ab^2cp\\ +b^4p^5+6b^4p^4+5b^4p^3-20b^4p^2\end{pmatrix}}{60a^2(1+p)^2}$$

We define the numerator as a quadratic function $w\left(b^2\right)$ of b^2.

$$w\left(b^2\right)=600a^2c^2p-120a^2c^2+40ab^2cp^3+160ab^2cp^2-120ab^2cp$$
$$+b^4p^5+6b^4p^4+5b^4p^3-20b^4p^2$$
$$=\left(p^5+6p^4+5p^3-20p^2\right)\left(b^2\right)^2+\left(40acp^3+160acp^2-120acp\right)b^2$$
$$+600a^2c^2p-120a^2c^2$$

Because $p^5+6p^4+5p^3-20p^2\le 0$ holds when $p\in[0,1]$, $w\left(b^2\right)$ reaches its maximum at

$$b^2=-\frac{40acp^3+160acp^2-120acp}{2\left(p^5+6p^4+5p^3-20p^2\right)}=\frac{20ac\left(3-p^2-4p\right)}{p^4+6p^3+5p^2-20p}.$$

$$\frac{20ac\left(3-p^2-4p\right)}{p^4+6p^3+5p^2-20p}-\frac{24acp}{3p+1-p^3-3p^2}$$
$$=ac\frac{4p^5+4p^4-40p^2-100p-60}{p^7+9p^6+20p^5-24p^4-81p^3+55p^2+20p}$$

Under $p\in[0,1]$, we find that $p^7+9p^6+20p^5-24p^4-81p^3+55p^2+20p>0$ and $4p^5+4p^4-40p^2-100p-60<0$, so, $\frac{20ac\left(3-p^2-4p\right)}{p^4+6p^3+5p^2-20p}<\frac{24acp}{-p^3-3p^2 3p+1}$ and $w\left(b^2\right)$ decreases under $\frac{24acp}{-p^3-3p^2 3p+1}<b^2\le 4ac$.

$$w\left(b^2\right)_{\max:p\in[0,1]} < w\left(\frac{24acp}{3p+1-p^3-3p^2}\right)$$

$$w\left(b^2 = \frac{24acp}{3p+1-p^3-3p^2}\right)$$
$$= -24a^2c^2\frac{-9p^7-9p^6+75p^5+115p^4+25p^3-15p^2+5p+5}{\left(-p^3-3p^2+3p+1\right)^2} < 0$$

Thus, we can prove that $w\left(b^2\right) < 0$ and $Var_{Interval} < Var_{Same}$ under case 2.

Proof of Theorem 2

From Figure 4, we can calculate productivity's mean and variance of production line by Equation 5 and 6.

$$E_{Interval} = \int_{T_i}^{s+T_b+T_i}\frac{1}{s+T_b}\left(E_1+E_2+E_3\right)dt$$
$$= \frac{1}{s+T_b}\left(\int_{T_i}^{s}E_1dt + \int_{T_i}^{s+T_i}E_2dt + \int_{s+T_b}^{s+T_b+T_i}E_3dt\right) = \frac{6ac-b^2}{3a(1+p)}$$

$$Var_{Interval} = \int_{T_i}^{s+T_b+T_i}\frac{1}{s+T_b}\left(E_1+E_2+E_3-\overline{E_{1,2,3}}\right)^2 dt$$
$$= \frac{1}{s+T_b}\left(\begin{aligned}&\int_{T_i}^{s}\left(E_1+E_2-\overline{E_{1,2,3}}\right)^2dt + \int_{s}^{s+T_b}\left(E_2-\overline{E_{1,2,3}}\right)^2dt\\ &+\int_{s+T_b}^{s+T_i}\left(E_2+E_3-\overline{E_{1,2,3}}\right)^2dt + \int_{s+T_i}^{s+T_i+T_b}\left(E_3-\overline{E_{1,2,3}}\right)^2dt\end{aligned}\right)$$

We derivate the variance $Var_{Interval}$ in the decision variable T_i.

$$\frac{\partial\left(Var_{Interval}\right)}{\partial T_i} =$$
$$-\frac{4a^2p+4a^2}{3p+3}T_i^3 + \left(\frac{4abp^2+8abp+4ab}{3p+3} + \frac{\left(2a^2p+2a^2\right)\left(b+bp\right)}{3a(1+p)}\right)T_i^2$$
$$+\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3p+3} - \frac{\left(b+bp\right)\left(2abp^2+4abp+2ab\right)}{3a(1+p)}\right)T_i$$
$$+\frac{\left(b+bp\right)\left(b^2p^3+3b^2p^2+12acp\right)}{3a(1+p)}$$
$$= -\frac{4a^2}{3}T_i^3 + 2ab(1+p)T_i^2$$
$$+\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3(p+1)} - \frac{2b^2(p+1)^2}{3}\right)T_i + \frac{b\left(b^2p^3+3b^2p^2+12acp\right)}{3a}$$

We solve the equation $\frac{\partial\left(Var_{Interval}\right)}{\partial T_i}=0$ to obtain three solutions $T_i=\frac{1}{2}\left(T_b+s\right)$ and $T_i=\frac{1}{2}\left(T_b+s\right)\pm\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}}$ .Let $A=-\frac{4}{3}a^2$, $B=2ab\left(1+p\right)$ and $C=-\frac{2b^2p^3+6b^2p^2+24acp}{3\left(p+1\right)}-\frac{2b^2\left(p+1\right)^2}{3}$, we can get the following equation to find the solution number of $\frac{\partial\left(Var_{Interval}\right)}{\partial T_i}=0$.

$$\Delta=4B^2-12AC$$

$$=4\left(2ab\left(1+p\right)\right)^2-12\left(-\frac{4a^2}{3}\right)\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3\left(p+1\right)}-\frac{2b^2\left(p+1\right)^2}{3}\right)$$

$$=16a^2\left(\frac{b^2\left(1+3p-p^3-3p^2\right)-24acp}{3\left(p+1\right)}\right)$$

$$\Delta=4B^2-12AC=0\Rightarrow b^2=\frac{24acp}{-p^3-3p^2+3p+1}$$

Because $p\in\left(0,1\right)$, thus, the denominator $-p^3-3p^2+3p+1>0$. The following conclusions are obtained: when $0<b^2\le\frac{24acp}{3p+1-p^3-3p^2}$, $\frac{\partial\left(Var_{Interval}\right)}{\partial T_i}$ is a monotonic function and $Var_{Interval}$ get the unique extremum at $T_i=\frac{1}{2}\left(T_b+s\right)$; when $\frac{24acp}{3p+1-p^3-3p^2}<b^2<4ac$, $\frac{\partial\left(Var_{Interval}\right)}{\partial T_i}$ has three monotonic intervals and its solutions can be obtained at $T_i=\frac{1}{2}\left(1+p\right)\frac{b}{a}\pm\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}}$ or $T_i=\frac{1}{2}\left(T_b+s\right)$. Additionally, other two extremum points is symmetrical about $T_i=\frac{1}{2}\left(T_b+s\right)$ and $Var_{Interval}$ gets the same value in those two points. In our context, the interval time between two production lines has special restriction, $T_i\in\left[T_b,s\right]$. Then, we must discuss whether all solutions of $\frac{\partial\left(Var_{Interval}\right)}{\partial T_i}$ fall into the range of $\left[T_b,s\right]$. We know that $\frac{1}{2}\left(T_b+s\right)$ is lie in the midpoint of $\left[T_b,s\right]$.

Let $T_i=\frac{1}{2}\left(T_b+s\right)+\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}}\le\frac{b}{a}$, we can get

$$\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}}\le\frac{1}{2}\left(1-p\right)\frac{b}{a}$$

$$\Rightarrow\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}\le\frac{1}{4}\left(1-p\right)^2\frac{b^2}{a^2}$$

$$\Rightarrow b^2\le\frac{24acp}{2p\left(2-p^2-p\right)}$$

Let $g\left(p\right)=\left(3p+1-p^3-3p^2\right)-2p\left(2-p^2-p\right)$

$$g\left(p\right)=\left(3p+1-p^3-3p^2\right)-2p\left(2-p^2-p\right)=\left(1-p\right)\left(1-p^2\right)\ge 0$$

$$\Rightarrow\frac{p}{3p+1-p^3-3p^2}\le\frac{p}{2p\left(2-p^2-p\right)}$$

$b^2>\frac{24acp}{2p\left(2-p^2-p\right)}$ does not exist because $b^2<4ac<\frac{24acp}{2p\left(2-p^2-p\right)}$. Therefore, there are three solutions when the condition $\frac{24acp}{-p^3-3p^2+3p+1}<b^2\le 4ac$ is satisfied.

Above all, we can get that, during the period $\left[T_b,s\right]$, the variance of production line $Var_{Interval}$ has only one extremum at $T_i=\frac{1}{2}\left(T_b+s\right)$ when the parameters satisfy the condition of $0<b^2\le\frac{24acp}{3p+1-p^3-3p^2}$; there are three extremums at $T_i=\frac{1}{2}\left(T_b+s\right)\pm\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2\left(p+1\right)}}$ and $T_i=\frac{1}{2}\left(T_b+s\right)$ when the parameters satisfy the condition $\frac{24acp}{3p+1-p^3-3p^2}<b^2\le 4ac$.

Proof of Theorem 3

Next, we focus on analyzing each theorem to capture the minimum variance of $Var_{Interval}$.

Firstly, we analyses the case that $Var_{Interval}$ have only one extremum at $T_i=\frac{1}{2}\left(T_b+s\right)$ with condition $0<b^2\le\frac{24acp}{3p+1-p^3-3p^2}$. We take the second derivative of Equation 6 with parameter T_i from equation.

$$\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}=-4a^2T_i^2+4ab\left(1+p\right)T_i+\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3\left(p+1\right)}-\frac{2b^2\left(p+1\right)^2}{3}\right)$$

$$\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(s+T_b)} = -4a^2\left(\frac{1}{2}(1+p)\frac{b}{a}\right)^2 + 4ab(1+p)\frac{1}{2}(1+p)\frac{b}{a}$$

$$+\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3(p+1)} - \frac{2b^2(p+1)^2}{3}\right)$$

$$= \frac{b^2\left(1-p^3+3p-3p^2\right)-24acp}{3(p+1)}$$

When $b^2 > \frac{24acp}{-p^3-3p^2+3p+1}$, we get $\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(s+T_b)} > 0$ (not exist); when $b^2 < \frac{24acp}{-p^3-3p^2+3p+1}$, we get $\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(s+T_b)} < 0$. $Var_{Interval}$ obtains its maximum at $T_i = \frac{1}{2}(T_b+s)$ when $0 < b^2 \le \frac{24acp}{-p^3-3p^2+3p+1}$, therefore, production system will get the minimum variability at $T_i = T_b$ or $T_i = s$ when $0 < b^2 \le \frac{24acp}{-p^3-3p^2+3p+1}$.

Then, we analyze the case that $Var_{Interval}$ have three extremums at $T_i = \frac{1}{2}(T_b+s)$ and $T_i = \frac{1}{2}(T_b+s) \pm \sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2(p+1)}}$ with condition $\frac{24acp}{-p^3-3p^2+3p+1} < b^2 \le 4ac$. We also take the second derivative of Equation 6 with parameter T_i. As for the extremum $T_i = \frac{1}{2}(T_b+s)$, we know that $\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(s+T_b)}$ must be larger than zero under this case. So, $T_i = \frac{1}{2}(T_b+s)$ is a minimum point. For other two extremum points, we have the following proof.

$$\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(1+p)\frac{b}{a}\pm\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2(p+1)}}} = -4a^2\left(\frac{1}{2}(1+p)\frac{b}{a} \pm \sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2(p+1)}}\right)^2$$

$$+4ab(1+p)\left(\frac{1}{2}(1+p)\frac{b}{a} \mp \sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2(p+1)}}\right)$$

$$+\left(-\frac{2b^2p^3+6b^2p^2+24acp}{3(p+1)} - \frac{2b^2(p+1)^2}{3}\right)$$

$$= \frac{2\left(-b^2\left(3p+1-p^3-3p^2\right)+24acp\right)}{3(p+1)}$$

Under the condition $\frac{24acp}{-p^3-3p^2+3p+1}<b^2\le 4ac$, inequality $\left.\frac{\partial^2\left(Var_{Interval}\right)}{\partial T_i^2}\right|_{T_i=\frac{1}{2}(1+p)\frac{b}{a}\pm\sqrt{\frac{\left(3p+1-p^3-3p^2\right)b^2-24acp}{4a^2(p+1)}}}<0$ holds and the above two points are both maximum points.

Above all, it is obvious that the minimum of $Var_{Interval}$ must be obtained at $T_i=\frac{1}{2}\left(T_b+s\right)$ or other two points ($T_i=T_b$ or $T_i=s$). Next, we discuss the difference between $Var_{Interval}$ and $Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}}$, $Var_{Interval_{T_i=p\frac{b}{a}}}$ or $Var_{Interval_{T_i=\frac{b}{a}}}$.

$$Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}}-Var_{Interval_{T_i=p\frac{b}{a}}}=Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}}-Var_{Interval_{T_i=\frac{b}{a}}}$$
$$=\frac{b^2\left(p-1\right)^2\left(3b^2p^3+5b^2p^2-7b^2p-b^2+48acp\right)}{48a^2\left(1+p\right)}$$
$$=\frac{b^2\left(p-1\right)^2\left(\left(3p^3+5p^2-7p-1\right)b^2+48acp\right)}{48a^2\left(1+p\right)}$$

Let $w\left(b^2\right)=\left(3p^3+5p^2-7p-1\right)b^2+48acp$, we know that $3p^3+5p^2-7p-1<0$ when $p\in(0,1)$, we can obtain the following inequality of $w\left(b^2\right)$: $w\left(b^2=4ac\right)\le w\left(b^2\right)<w\left(b^2=\frac{24acp}{3p+1-p^3-3p^2}\right)$.

$$w(4ac)=\left(3p^3+5p^2-7p-1\right)4ac+48acp$$
$$=4ac\left(3p^3+5p^2+5p-1\right)$$

When $3p^3+5p^2+5p-1>0$, $w(4ac)>0$, otherwise, $w(4ac)\le 0$.

$$w\left(\frac{24acp}{3p+1-p^3-3p^2}\right)=\left(3p^3+5p^2-7p-1\right)\frac{24acp}{3p+1-p^3-3p^2}+48acp$$
$$=24acp\frac{p^3-p^2-p+1}{3p+1-p^3-3p^2}>0$$

Thus, when $3p^3+5p^2+5p-1>0$, $0\le w(4ac)<w\left(b^2\right)$; When $3p^3+5p^2+5p-1\le 0$, $w\left(b^2\right)=0$ will have a solution $b^2=\frac{48acp}{-3p^3-5p^2+7p+1}$ which falls into the range of $\left(\frac{24acp}{3p+1-p^3-3p^2},4ac\right)$. In the end, we can have the conclusions: when $3p^3+5p^2+5p-1>0$ or $3p^3+5p^2+5p-1\le 0$

and $\frac{24acp}{3p+1-p^3-3p^2} < b^2 < \frac{48acp}{-3p^3-5p^2+7p+1}$, $w(b^2) > 0$, that is to say $Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}} > Var_{Interval_{T_i=p\frac{b}{a}}}$; when $3p^3+5p^2+5p-1 \le 0$ and $\frac{48acp}{-3p^3-5p^2+7p+1} < b^2 \le 4ac$, $w(b^2) < 0$, that is to say $Var_{Interval_{T_i=\frac{1}{2}(p+1)\frac{b}{a}}} < Var_{Interval_{T_i=p\frac{b}{a}}}$.

In summary, under $T_i \in [T_b, s]$, $Var_{Interval}$ will get the minimum at $T_i = \frac{1}{2}(T_b + s)$ when $3p^3+5p^2+5p-1 \le 0$ and $\frac{48acp}{-3p^3-5p^2+7p+1} < b^2 \le \frac{24acp}{2p(2-p^2-p)}$; $Var_{Interval}$ will get the minimum at $T_i = T_b$ or $T_i = s$ when $3p^3+5p^2+5p-1 > 0$ or $3p^3+5p^2+5p-1 \le 0$ and $\frac{24acp}{-p^3-3p^2+3p+1} < b^2 < \frac{48acp}{-3p^3-5p^2+7p+1}$ or $0 < b^2 \le \frac{24acp}{-p^3-3p^2+p+1}$.