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Solution Of A System Of Two Partial Differential Equations Of The Second Order Using Two Series

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Abstract

For a system of two partial differential equations of second order, we obtain and justify two asymptotics solutions in the form of two series with respect to the small parameter ε . We have proven the solution is unique and uniform in the domain Ω , and, further, each the asymptotics approximations are within $O(\varepsilon^{n+1})$.

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Introduction

Levenshtam (2009), considered systems of ordinary differential equations of the first order whose terms oscillate with a high frequency ω . For the problem on periodic solutions, the author justifies the averaging method and establishes a posteriori bounds for the error of partial sums of the complete asymptotic expansion for the solution. Asymptotic expansions of periodic solutions of second –and third- order equations and formal asymptotics of such solutions in the case of equations of arbitrary order were constructed in Abood^{1,2} (2004). The first-order asymptotic form was obtained and proved for the solution of a system of two partial differential equations with small parameters in the derivatives for the regular part, two boundary-layer parts and corner boundary part in Vasil'eva and Butuzob (1990). In Levenshtam and Abood (2005), an algorithm of asymptotic integration of the initial-boundary-value problem for the heat-conduction equation with minor terms (nonlinear sources of heat) in a thin rod of thickness $\varepsilon = \omega^{-1/2}$ oscillating in time with frequency ω^{-1} was proposed. In the present paper, we consider a system of partial differential equations of the second order and solve this system with the aid of two series and some conditions. Our paper continues the line of research initiated in Levenshtam and Abood (2005). Examples of applications of systems of second order partial differential equations in modelling can be found in elasticity in Nerantzaki and Kandilas (2008) and packed-bed electrode in Xiao-Ying Qin and Yan-Ping Sun (2009).

Statement Of The Problem

We study a system of second order partial differential equations with initial-boundary-value conditions

$$\begin{aligned} \varepsilon \left(\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial y^2} \right) + b_1(x) \frac{\partial u}{\partial x} &= a_1(x, t)u + \sum_{i=0}^{\infty} \varepsilon^i f_{1i}(u, x, y, t), \\ \frac{\partial v}{\partial t} + \varepsilon \left(\frac{\partial^2 v}{\partial y^2} + b_2(x) \frac{\partial v}{\partial x} \right) &= a_2(x, t)v + \sum_{i=0}^{\infty} \varepsilon^i f_{2i}(u, x, y, t), \end{aligned} \tag{2.1}$$

in the domain $(x, y, t) \in \Omega = (0 \leq x \leq 1) \times (0 \leq y \leq 1) \times (0 \leq t < T)$.

The initial boundary conditions are

$$\begin{aligned}
 u|_{t=0} = 0, \quad u|_{x=0,1} = 0, \quad v|_{t=0} = 0, \quad v|_{x=0,1} = 0, \\
 \left. \frac{\partial u}{\partial y} \right|_{y=0,1} = 0, \quad \left. \frac{\partial v}{\partial y} \right|_{y=0,1} = 0.
 \end{aligned}
 \tag{2.2}$$

where ε is a small parameter, and the functions $f_1(u, x, y, t, \varepsilon)$, and $f_2(u, x, y, t, \varepsilon)$ are continuous and infinitely differentiable with respect to each of their arguments. The construction is made under the following conditions.

I. The functions $b_i(x), a_i(x, t)$ and $f_i, i = 1, 2$ have continuous derivatives of order $(n + 2)$.

II. We can assume that $b_2 = 1$. If $b_2 = b_2(x)$, then making the change of the variable $z = \int_0^x b_2^{-1}(\sigma) d\sigma$, and we shall assume that $b_1 = b(x) > 0$ and $b_2 = 1$.

Algorithm For Construction Of The Asymptotics

We seek an asymptotics expansion of the solution of problem (2.1) and (2.2) as the following two series in the powers of ε in the form

$$\begin{aligned}
 u(x, y, t, \varepsilon) &= \sum_{i=0}^{\infty} \varepsilon^i [\bar{u}_i(x, y, t) + p_i u(x, y, \tau) + q_i u(\xi, y, t)], \\
 v(x, y, t, \varepsilon) &= \sum_{i=0}^{\infty} \varepsilon^i [\bar{v}_i(x, y, t) + p_i v(x, y, \tau) + q_i v(\xi, y, t)],
 \end{aligned}
 \tag{3.1}$$

where \bar{u}_i and \bar{v}_i are coefficients of the regular part of the asymptotic, $p_i u, p_i v, q_i u$ and $q_i v$ are boundary-layer functions describing boundary layers near the initial instant of time $t = 0$ and the ends of the rod $x = 0$ and $x = 1$. The boundary-layer variables are

$$\tau = \frac{t}{\varepsilon}, \text{ and } \xi = \frac{x}{\varepsilon}.$$

Regular Parts Of The Asymptotics \bar{u} And \bar{v}

We substitute series (3.1) into equations (2.1) - (2.2) and the coefficients of the same powers of ε in the left and right-hand sides of the obtained relations are equated. If the variable τ is assumed to be independent of t and functions depending on ε are represented by the corresponding asymptotic series then problem (2.1) and (2.2) becomes

$$\begin{aligned}
 \varepsilon \left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) + b_1(x) \frac{\partial \bar{u}}{\partial x} &= a_1(x, t) \bar{u} + \sum_{i=0}^{\infty} \varepsilon^i f_{1i}(\bar{u}, x, y, t), \\
 \frac{\partial \bar{v}}{\partial t} + \varepsilon \left(\frac{\partial^2 \bar{v}}{\partial y^2} + b_2(x) \frac{\partial \bar{v}}{\partial x} \right) &= a_2(x, t) \bar{v} + \sum_{i=0}^{\infty} \varepsilon^i f_{2i}(\bar{v}, x, y, t), \\
 \bar{u} \Big|_{t, \tau=0} = 0, \quad \bar{u} \Big|_{x=0,1} = 0, \quad \bar{v} \Big|_{t, \tau=0} = 0, \quad \bar{v} \Big|_{x=0,1} = 0, \\
 \frac{\partial \bar{u}}{\partial y} \Big|_{y=0,1} = 0, \quad \frac{\partial \bar{v}}{\partial y} \Big|_{y=0,1} = 0.
 \end{aligned}
 \tag{4.1}$$

The following problems for the regular coefficients u_0 and v_0 are obtained:

$$\begin{aligned}
 b(x) \frac{\partial \bar{u}_0}{\partial x} &= a_1(x, t) \bar{u}_0 + f_{10}(\bar{u}_0, x, y, t), \\
 \frac{\partial \bar{v}_0}{\partial x} &= a_2(x, t) \bar{v}_0 + f_{20}(\bar{v}_0, x, y, t), \\
 \bar{u}_0(0, y, t) = 0, \quad \bar{v}_0(x, y, 0) = 0, \\
 \frac{\partial \bar{u}_0}{\partial y} \Big|_{y=0,1} = 0, \quad \frac{\partial \bar{v}_0}{\partial y} \Big|_{y=0,1} = 0,
 \end{aligned}
 \tag{4.2}$$

By direct integration, it can be seen [Viik (2010), Viika and Rõõm (2008), Tokovyy and Chien-Ching Ma,(2009), Toshiki, Son Shin, Murakami and Ngoc(2007)] system (4.2) is equivalent to system of integral equations

$$\bar{u}_0(x, t) = \int_0^x \exp\left(\int_{\sigma}^x b^{-1}(p) a_1(p, t) dp\right) b^{-1}(\sigma) [\bar{v}_0(\sigma, t) + f_{10}(\sigma, y, t)] d\sigma,
 \tag{4.3}$$

$$\bar{v}_0(x, t) = \int_0^t \exp\left(\int_s^t a_2(x, p) dp\right) [\bar{u}_0(x, s) + f_{20}(x, y, t)] ds.
 \tag{4.4}$$

Substituting (4.4) in (4.3), we arrive at an integral equation with respect to $\bar{u}_0(x, t)$:

$$\bar{u}_0(x, t) = \int_0^t \int_0^x G_0(x, t, \sigma, s) \bar{u}_0(\sigma, s) d\sigma ds + g_0(x, t),$$

where $G_0(x, t, \sigma, s)$ and $g_0(x, t)$ are known functions. The solution of this integral equation can be expressed in terms of the resolvent $K_0(x, t, \sigma, s)$ of the Kernel $G_0(x, t, \sigma, s)$ (as in [Viik (2010), Viika and Rõõm (2008), Tokovyy and Chien-Ching Ma,(2009), Toshiki, Son Shin, Murakami and Ngoc(2007)])

$$\bar{u}_0(x, t) = g_0(x, t) + \int_0^t \int_0^x K_0(x, t, \sigma, s) g_0(\sigma, s) d\sigma ds.$$

After that, the function $\bar{v}_0(x, t)$ can be determined by (4.4).

The system of equations for \bar{u}_1 and \bar{v}_1 is

$$\begin{aligned}
 b(x) \frac{\partial \bar{u}_1}{\partial x} &= a_1(x, t) \bar{u}_1(x, t) + \frac{\partial}{\partial u} f_{10}(\bar{u}_0 x, y, t) \bar{u}_0 + f_{11}(\bar{u}_0 x, y, t) - \\
 &\quad \frac{\partial \bar{u}_0}{\partial t} - \frac{\partial^2 \bar{u}_0}{\partial y^2}, \\
 \frac{\partial \bar{v}_1}{\partial t} &= a_2(x, t) \bar{v}_1(x, t) + \frac{\partial}{\partial v} f_{20}(\bar{v}_0 x, y, t) \bar{u}_0 + f_{21}(\bar{v}_0 x, y, t) - \\
 &\quad \frac{\partial^2 \bar{u}_0}{\partial y^2} - \frac{\partial \bar{v}_0}{\partial x}, \\
 \bar{u}_1(0, y, t) &= -q_1 u(0, y, t), \quad \bar{v}_1(x, y, 0) = -p_1 v(x, y, 0), \\
 \frac{\partial \bar{u}_1}{\partial y} \Big|_{y=0,1} &= 0, \quad \frac{\partial \bar{v}_1}{\partial y} \Big|_{y=0,1} = 0.
 \end{aligned} \tag{4.5}$$

From system (4.5) and direct integration, we have the system of integral equations

$$\begin{aligned}
 \bar{u}_1(x, t) &= \int_0^x \exp\left\{ \int_s^x a_1(p, t) dp \right\} b^{-1}(s) [a_2(s, t) \bar{v}_1(s, t) + \\
 &\quad \frac{\partial}{\partial u} f_{10}(\bar{u}_0 x, y, t) \bar{u}_0 + f_{11}(\bar{u}_0 x, y, t) - \frac{\partial \bar{u}_0}{\partial t} - \frac{\partial^2 \bar{u}_0}{\partial y^2}] ds,
 \end{aligned} \tag{4.6}$$

$$\begin{aligned}
 \bar{v}_1(x, t) &= \int_0^t \exp\left\{ \int_s^t a_2(x, p) dp \right\} [\bar{a}_1(x, s) \bar{u}_1(x, s) + \\
 &\quad + \frac{\partial}{\partial v} f_{20}(\bar{v}_0 x, y, t) \bar{v}_0 + f_{21}(\bar{v}_0 x, y, t) - \frac{\partial^2 \bar{u}_0}{\partial y^2} - \frac{\partial \bar{v}_0}{\partial x}] ds.
 \end{aligned} \tag{4.7}$$

Substituting (4.7) in (4.6), we arrive at the integral equation with respect to $\bar{u}_1(x, t)$:

$$\bar{u}_1(x, t) = \int_0^x \int_0^t G_1(x, y, t, \sigma, s) \bar{u}_1(\sigma, s) d\sigma ds + g_1(x, y, t),$$

where $G_1(x, y, t, \sigma, s)$ and $g_1(x, y, t)$ are known functions. The solution of this integral equation can be expressed in terms of the resolvent $K_1(x, y, t, \sigma, s)$ of the Kernel $G_1(x, y, t, \sigma, s)$ [Viik (2010), Viika and Rõõm (2008), Tokovyy and Chien-Ching Ma, (2009), Toshiki, Son Shin, Murakami and Ngoc (2007)] and we have

$$\bar{u}_1(x, y, t) = g_1(x, y, t) + \int_0^x \int_0^t K_1(x, y, \sigma, s) g_1(\sigma, s) d\sigma ds. \tag{4.7).}$$

For regular terms \bar{u}_n and \bar{v}_n we obtain the problem

$$\begin{aligned}
 b(x) \frac{\partial \bar{u}_n}{\partial x} &= a_1(x, t) \bar{u}_n(x, t) + i_n(\bar{u}_0 x, y, t), \\
 \bar{u}_n(0, y, t) &= -q_n u(0, y, t), \quad \left. \frac{\partial \bar{u}_n}{\partial y} \right|_{y=0,1} = 0.
 \end{aligned}
 \tag{4.8}$$

$$\begin{aligned}
 \frac{\partial \bar{v}_n}{\partial t} &= a_2(x, t) \bar{v}_n(x, t) + \bar{i}_n(\bar{v}_0 x, y, t), \\
 \bar{v}_n(x, y, 0) &= -p_n v(x, y, 0), \quad \left. \frac{\partial \bar{v}_n}{\partial y} \right|_{y=0,1} = 0.
 \end{aligned}
 \tag{4.9}$$

where $i_n(\bar{u}_0 x, y, t) = \sum_{k=0}^n \frac{1}{k!} \frac{\partial^k}{\partial \bar{u}^k} f_{1k}(\bar{u}_0 x, y, t) \bar{u}_k(x, t) - \frac{\partial \bar{u}_{n-1}}{\partial t} - \frac{\partial^2 \bar{u}_{n-1}}{\partial y^2}$ and

$\bar{i}_n(\bar{v}_0 x, y, t) = \sum_{k=0}^n \frac{1}{k!} \frac{\partial^k}{\partial \bar{u}^k} f_{2k}(\bar{v}_0 x, y, t) \bar{v}_k(x, t) - \frac{\partial^2 \bar{u}_0}{\partial y^2} - \frac{\partial \bar{v}_0}{\partial x}$. This problem is quite similar to problem (4.5) and can be solved by reduction to a system of integral equations.

Boundary Parts Of The Asymptotics pu And pv

We construct the following group of coefficients of two series (3.1) that is the boundary-layer parts $pu(x, y, \tau, \varepsilon)$ and $pv(x, y, \tau, \varepsilon)$. We consider problem (2.1) in a neighbourhood of the upper boundary ($t = 0$) of the domain Ω and perform the change of variables $t = \varepsilon \tau$; we obtain the system

$$\begin{aligned}
 \frac{\partial pu}{\partial \tau} + \varepsilon \frac{\partial^2 pu}{\partial y^2} + b(x) \frac{\partial pu}{\partial x} &= a_1(x, t) pu + f_1(pu, x, y, \varepsilon \tau, \varepsilon) \\
 \frac{\partial pv}{\partial \tau} + \varepsilon \left(\frac{\partial^2 pv}{\partial y^2} + \frac{\partial pv}{\partial x} \right) &= a_2(x, t) pv + f_2(pv, x, y, \varepsilon \tau, \varepsilon),
 \end{aligned}
 \tag{5.1}$$

$$pu(x, y, 0) = -\bar{u}(x, y, 0), \quad pv(x, y, 0) = -\bar{v}(x, y, 0), \quad \left. \frac{dpu}{dy} \right|_{y=0,1} = 0, \quad \left. \frac{dpv}{dy} \right|_{y=0,1} = 0.$$

The boundary-layer functions are p_0u and p_0v . In this case $p_0v \equiv 0$.

For p_0u we obtain the problem

$$\begin{aligned}
 \frac{\partial p_0u}{\partial \tau} + b(x) \frac{\partial p_0u}{\partial x} &= a_1(x, 0) p_0u, \quad 0 \leq x \leq X, \quad \tau \geq 0, \\
 p_0u(0, y, \tau) &= 0, \quad p_0u(x, y, 0) = -\bar{u}_0(x, y, 0), \\
 \left. \frac{dp_0u}{dy} \right|_{y=0,1} &= 0, \quad \left. \frac{dp_0v}{dy} \right|_{y=0,1} = 0.
 \end{aligned}
 \tag{5.2}$$

By direct integration, equation (5.2) has a classical solution

$$p_0 u(x, \tau) = \begin{cases} -\bar{u}_0(\theta^{-1}(\theta(x) - \tau), 0) \exp\left(\int_0^\tau a_1(\theta^{-1}(\theta(x) - \tau + s), 0) ds\right), & 0 \leq \tau \leq \theta \\ 0, & \tau \geq \theta, \end{cases}$$

where $\theta(x) = \int_0^x b^{-1}(\sigma) d\sigma$ and $\theta^{-1}(z)$ is the function inverse to $z = \theta(x)$.

The system of equations for $p_1 u$ and $p_1 v$ is

$$\begin{aligned} \frac{\partial p_1 u}{\partial \tau} + b(x) \frac{\partial p_1 u}{\partial x} &= a_1(x, t) p_1 u + \Delta_1(x, \tau) \\ p_1 u(x, y, 0) &= -\bar{u}_1(x, y, 0), \quad p_1 u(0, y, \tau) = 0, \\ \left. \frac{dp_1 u}{dy} \right|_{y=0,1} &= 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial p_1 v}{\partial x} &= a_2(x, t) p_1 v + \bar{\Delta}_1(x, \tau), \\ p_1 v(x, y, 0) &= -\bar{v}_1(x, y, 0) \\ \left. \frac{dp_1 v}{dy} \right|_{y=0,1} &= 0. \end{aligned}$$

Here $\Delta_1(x, \tau) = \frac{\partial a_1}{\partial t}(x, 0) \tau p_0 u(x, \tau) + f_{10}(p_0 u, x, y, 0, 0) - \frac{\partial^2 p_0 u}{\partial y^2}$ and

$\bar{\Delta}_1(x, \tau) = \frac{\partial a_2}{\partial t}(x, 0) \tau p_0 v(x, \tau) + f_{20}(p_0 v, x, y, 0, 0) - \frac{\partial^2 p_1 v}{\partial y^2}$ is smooth. In the

same way we obtain

$$p_1 v = a_2(x, 0) \int_{\theta(x)}^\tau p_0 v(x, s) ds \begin{cases} a_2(x, 0) \int_{\theta(x)}^\tau p_0 v(x, s) ds, & 0 \leq \tau \leq \theta(x) \\ 0, & \tau \geq \theta(x). \end{cases}$$

$$p_1 u = \begin{cases} -\bar{u}_1(\theta^{-1}(\theta(x) - \tau), 0) \exp\left(\int_0^\tau a_1(\theta^{-1}(\theta(x) - \tau + s), 0) ds\right) \times \\ \Delta_1(\theta^{-1}(\theta(x) - \tau + s), 0) ds, & 0 \leq \tau \leq \theta(x) \\ 0, & \tau \geq \theta(x). \end{cases}$$

The function $p_n u$ can be defined as the solution of the problems

$$\frac{\partial p_n u}{\partial \tau} + b(x) \frac{\partial p_n u}{\partial x} = a_1(x, t) p_n u + \Delta_n(x, y, \tau), \tag{5.3}$$

$$p_n u(x, y, 0) = -\bar{u}_n(x, y, 0), \quad p_n u(0, y, \tau) = 0, \tag{5.4}$$

$$\left. \frac{dp_n u}{dy} \right|_{y=0,1} = 0,$$

where

$$\Delta_n(x, y, \tau) = \sum_{k=0}^n \frac{1}{k!} \tau^k \left[\frac{\partial a_{1k}}{\partial t}(x, 0) p_{k-1} u(x, \tau) + f_{1k}(p_0 u, x, y, 0, 0) + \frac{\partial^k}{\partial^k p_0 u} f_{1k}(p_0 u, x, y, 0, 0) \right] - \frac{\partial^2 p_{n-1} u}{\partial y^2},$$

since (5.3) is a partial differential equation of the first order and first degree. Here the smooth function $\Delta_n(x, y, \tau)$ is known. Using the initial and boundary conditions (5.4), we obtain

$$p_n u = \begin{cases} \left[-\bar{u}_n(\theta^{-1}(\theta(x) - \tau), 0) \exp\left(\int_0^\tau a_1(\theta^{-1}(\theta(x) - \tau + s), 0) ds\right) + \int_0^\tau \Delta_n(\theta^{-1}(\theta(x) - \tau + s), 0) ds \right] \times \\ \quad \Delta_n(\theta^{-1}(\theta(x) - \tau + s), 0) ds, \quad 0 \leq \tau \leq \theta(x) \\ 0, \quad \tau \geq \theta(x). \end{cases}$$

For the function $p_n v$, it is the solution of the problem

$$\frac{\partial p_n v}{\partial \tau} = \bar{\Delta}_n(x, y, \tau),$$

$$p_n v(x, y, 0) = -\bar{v}(x, y, 0)$$

$$\left. \frac{dp_n v}{dy} \right|_{y=0,1} = 0,$$

where

$$\bar{\Delta}_n(x, y, \tau) = a_2(x, t) p_n v + \sum_{k=0}^n \frac{1}{k!} \tau^k \left[\frac{\partial a_{2k}}{\partial t}(x, 0) p_{k-1} v(x, \tau) + f_{1k}(p_0 v, x, y, 0, 0) + \frac{\partial^k}{\partial^k p_0 v} f_{1k}(p_0 v, x, y, 0, 0) \right] - \frac{\partial p_{n-1} v}{\partial x} - \frac{\partial^2 p_{n-1} v}{\partial y^2},$$

that satisfies the condition $p_n(x, y, \infty) = 0$, where $\bar{\Delta}_n(x, y, \tau)$ is a known continuous function vanishing for $\tau \geq \theta(x)$ and with partial derivative making jump on the characteristic line $\tau = \theta(x)$. Integrating, we obtain

$$p_n v(x, y, \tau) = \begin{cases} \int_{\theta(x)}^{\tau} \bar{\Delta}_n(x, y, r) dr, & 0 \leq \tau \leq \theta(x), \\ 0, & \tau \geq \theta(x), \end{cases}$$

where the function $p_n v(x, y, \tau)$ is smooth everywhere.

Boundary Parts Of The Asymptotics qu And qv

Now, we find the coefficients $q_i u(\xi, y, t)$ and $q_i v(\xi, y, t)$. Consider problem (2.1) and the conditions (2.2) in a neighbourhood of the left boundary of the domain Ω and perform the change of variable $\xi = x\varepsilon^{-1}$. We obtain the system

$$\begin{aligned} \varepsilon \left(\frac{\partial qu}{\partial t} + \frac{\partial^2 qu}{\partial y^2} \right) + \frac{1}{\varepsilon} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon^k b(0) \xi^k \right) \frac{\partial qu}{\partial \xi} &= a_1(\varepsilon \xi, t) qu + f_1(qu, \varepsilon \xi, y, t, \varepsilon) \\ \frac{\partial qv}{\partial t} + \varepsilon \frac{\partial^2 qv}{\partial y^2} + \frac{\partial qv}{\partial \xi} &= a_2(\varepsilon \xi, t) qv(\xi, y, t, \varepsilon) + f_2(qu, \varepsilon \xi, y, t, \varepsilon). \end{aligned}$$

For $q_0 u(\xi, y, t)$ we obtain the equation $b(0) \frac{\partial q_0 u}{\partial \xi} = 0$.

From this equation, taking into account the standard condition for boundary parts at infinity, that means $q_0 u(\infty, t) = 0$, and so we have $q_0 u(\xi, y, t) \equiv 0$.

Now, for $q_0 v(\xi, y, t)$ we have the problem

$$\begin{aligned} \frac{\partial q_0 v}{\partial t} + \frac{\partial q_0 v}{\partial \xi} &= a_2(0, t) q_0 v(\xi, y, t), \quad \xi \geq 0, \quad 0 \leq t \leq T, \\ q_0 v(0, y, t) &= -\bar{v}_0(0, y, t), \quad q_0 v(\xi, y, 0) = 0, \\ \left. \frac{\partial q_0 v}{\partial y} \right|_{y=0,1} &= 0 \end{aligned}$$

Since this is a partial differential of first order and first degree, it's solution have the form

$$q_0 v(\xi, y, t) = \begin{cases} -\bar{v}_0(0, y, t - \xi) \exp \left(\int_0^{\xi} a_2(0, y, s + t - \xi) ds \right), & 0 \leq \xi \leq t \leq T \\ 0, & \xi \geq t. \end{cases}$$

It's possible to find q_1u and q_1v in the same way. Finally, we will continue to find q_nu and q_nv .

For q_nu we have the equation

$$b(0) \frac{\partial q_n u}{\partial \xi} = \Psi_n(\xi, y, t), \tag{6.1}$$

where

$$\Psi_n(\xi, y, t) = \sum_{k=0}^n \frac{1}{k!} \xi^k \left[\frac{\partial a_{1k}}{\partial t}(0, y, t) q_{k-1}u + f_{1k}(q_0u, 0, y, t, 0) + \frac{\partial^k}{\partial^k q_0u} f_{1k}(q_0u, x, y, 0, 0) \right] - \frac{\partial q_{n-1}u}{\partial t} \frac{\partial^2 q_{n-1}u}{\partial y^2},$$

is known and continuous everywhere.

The function $\Psi_n(\xi, y, t) = 0$ for $\xi \geq t$ (is evident from our findings in the above), so we can seek this function in the form

$$\Psi_n(\xi, y, t) = z(y)(t - \xi)Z_n(\xi, t), \text{ for } \xi \leq t. \tag{6.2}$$

The function $Z_n(\xi, t)$ is smooth everywhere. By the condition $q_nu(\infty, y, t) = 0$ and integrating (6.1), we have

$$q_nu(\xi, y, t) = \begin{cases} b^{-1}(0)z(y) \int_t^\xi (t-s)Z_n(s, t) ds, & 0 \leq \xi \leq t \leq T \\ 0, & \xi \geq t. \end{cases}$$

We can define the part $q_nv(\xi, y, t)$ as the solution of the problem

$$\frac{\partial q_nv}{\partial t} + \frac{\partial q_nv}{\partial \xi} = \bar{\Psi}_n(\xi, y, t), \tag{6.3}$$

$$q_nv(0, y, t) = -\bar{v}_n(0, y, t), \quad q_nv(\xi, y, 0) = 0,$$

$$\left. \frac{\partial q_nv}{\partial y} \right|_{y=0,1} = 0 \tag{6.4}$$

here

$$\bar{\Psi}_n(\xi, y, t) = a_2(0, y, t) p_n v + \sum_{k=0}^n \frac{1}{k!} \xi^k \left[\frac{\partial a_{2k}}{\partial t}(x, 0) p_{k-1} v(0, y, t) + f_{1k}(p_0 v, 0, y, t, 0) + \right. \\ \left. \frac{\partial^k}{\partial^k p_0 v} f_{1k}(p_0 v, 0, y, t, 0) \right] - \frac{\partial p_{n-1} v}{\partial x} - \frac{\partial^2 p_{n-1} v}{\partial y^2}. \quad \text{Also}$$

$\bar{\Psi}_n(\xi, y, t)$ is a known continuous function that satisfies the condition $p_n v(\infty, y, t) = 0$, vanishing for $\xi \geq t$ and with partial derivative making jump on the characteristic line $\xi = t$. Solving (6.3) and using (6.4), we obtain

$$q_n v(\xi, y, t) = \begin{cases} -\bar{v}_n(0, y, t - \xi) \exp\left(\int_0^\xi \bar{\Psi}_n(0, y, s + t - \xi) ds\right), & 0 \leq \xi \leq t \leq T \\ 0, & \xi \geq t. \end{cases}$$

where the function $q_n v(\xi, y, t)$ is smooth everywhere.

7. JUSTIFICATION OF THE ASYMOTOTICS

By $U_n(x, y, t, \varepsilon)$ and $V_n(x, y, t, \varepsilon)$ we denote the n -th partial sums of the series (3.1).

Theorem. *The solution $u(x, y, t, \varepsilon), v(x, y, t, \varepsilon)$ of problem (2.1) and (2.2) admits the asymptotic solution*

$$u(x, y, t, \varepsilon) - U_n(x, y, t, \varepsilon) = O(\varepsilon^{n+1}), \quad v(x, y, t, \varepsilon) - V_n(x, y, t, \varepsilon) = O(\varepsilon^{n+1}), \quad (7.1)$$

uniformly in the domain $\Omega = (0 \leq x \leq 1) \times (0 \leq y \leq 1) \times (0 \leq t \leq T)$.

Proof. We set $u = U_{n+1} + w_1$ and $v = V_{n+1} + w_2$. Substituting this in (1.1) and (2.1) we obtain the following problem for the remainder terms w_1 and w_2

$$\varepsilon \left(\frac{\partial w_1}{\partial t} + \frac{\partial^2 w_1}{\partial y^2} \right) + b_1(x) \frac{\partial w_1}{\partial x} = a_1(x, t) w_1 + \Pi_1(x, y, t, \varepsilon), \\ \frac{\partial w_2}{\partial t} + \varepsilon \left(\frac{\partial^2 w_2}{\partial y^2} + b_2(x) \frac{\partial w_2}{\partial x} \right) = a_2(x, t) w_2 + \Pi_2(x, y, t, \varepsilon), \\ w_1|_{r=0} = 0, \quad w_1|_{x=0,1} = 0, \quad w_2|_{r=0} = 0, \quad w_2|_{x=0,1} = 0, \\ \left. \frac{\partial w_1}{\partial y} \right|_{y=0,1} = 0, \quad \left. \frac{\partial w_2}{\partial y} \right|_{y=0,1} = 0.$$

Obviously, the inhomogeneous terms Π_1 and Π_2 can be estimated as $H_i(x, y, t, \varepsilon) = O(\varepsilon^{n+1})$ uniformly in Ω . We shall prove that the same estimate is valid for w_1 and w_2 . In view of the equalities

$$u - U_n = (u - U_{n+1}) + (U_{n+1} - U_n) = w_1 + O(\varepsilon^{n+1}), \\ v - V_n = w_2 + O(\varepsilon^{n+1}),$$

this will directly imply the assertion of the theorem.

We make the change of the variables $w_i = r_i e^{k(x+y+t)}$ ($i = 1, 2$), where $k - \text{const} > 0$. We obtain the following system of equations for r_1 and r_2 :

$$\begin{aligned} \varepsilon \left(\frac{\partial r_1}{\partial t} + \frac{\partial^2 r_1}{\partial y^2} \right) + b_1(x) \frac{\partial r_1}{\partial x} &= c_1(x, t) r_1 + \hat{\Pi}_1(x, y, t, \varepsilon), \\ \frac{\partial r_2}{\partial t} + \varepsilon \left(\frac{\partial^2 r_2}{\partial y^2} + b_2(x) \frac{\partial r_2}{\partial x} \right) &= c_2(x, t) r_2 + \hat{\Pi}_2(x, y, t, \varepsilon), \end{aligned} \tag{7.2}$$

with the homogeneous boundary conditions, as in the case of $r_i, i = 1, 2$. Here $c_1 = a_1(x, t) - k(b(x) + \varepsilon)$, $c_2 = a_2(x, t) - k(1 + \varepsilon)$, $\hat{\Pi}_i = \Pi_i e^{-k(x+t)} = O(\varepsilon^{n+1})$. Assume that $|r_1|$ has a maximum at a point $K_1(x_1, t_1)$ of Ω and $|r_2|$ has a maximum at a point $K_2(x_2, t_2)$, and assume that $|r_1(K_1)| \geq |r_2(K_2)|$. We shall consider the first equation of (7.2) at K_1 , (If $|r_1(K_1)| \leq |r_2(K_2)|$, then we shall consider the first equation of (7.2) at K_2). We rewrite this equation as

$$\varepsilon \left(\frac{\partial r_1}{\partial t} + \frac{\partial^2 r_1}{\partial y^2} \right) + b_1(x) \frac{\partial r_1}{\partial x} = c_1(x, t) r_1 + \hat{\Pi}_1(x, y, t, \varepsilon), \tag{7.3}$$

Assume that the function r_1 is negative and has a minimum at K_1 . (The case of a positive maximum can be considered in the same way.) Then $\frac{\partial r_1}{\partial t} \leq 0$ and $\frac{\partial r_1}{\partial x} \leq 0$ at this point (the inequality sign is possible only if K_1 lies on the boundary of Ω). Hence the left-hand side of (7.3) is negative at K_1 and is not larger than $r_1(K_1)$, while the right-hand side is of order ε^{n+1} .

Consequently, $|r_1(K_1)| = \max_{\Omega} |r_1(x, t)| = O(\varepsilon^{n+1})$. Since $|r_1(K_1)| \geq |r_2(K_2)|$, it follows that $|r_2(K_2)| = \max_{\Omega} |r_2(x, t)| = O(\varepsilon^{n+1})$. Therefore $r_i(x, t) = O(\varepsilon^{n+1})$ uniformly in Ω . Hence we also have $w_i = r_i e^{k(x+t)} = O(\varepsilon^{n+1})$ uniformly in G .

The proof of Theorem is complete.

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