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JOINT DISTRIBUTION OF MINIMUM OF N IID EXPONENTIAL RANDOM VARIABLES AND POISSON MARGINAL

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ABSTRACT

We introduced a random vector (X, N), where N has Poisson distribution and X are minimum of N independent and identically distributed exponential random variables. We present fundamental properties of this vector such as PDF, CDF and stochastic representations. Our results include explicit formulas for marginal and conditional distributions, moments and moments generating functions. We also derive moments estimators and maximum likelihood estimators of the parameter of this distribution.

Keywords: Hierarchical approach, Joint distribution, Poisson marginal, Moments estimators, Maximum likelihood estimators, Marginal distributions, Conditional distributions, Stochastic representations.

INTRODUCTION

We suppose that $X_1, X_2, X_3, \dots, X_N$ are independent and identically distributed (IID) exponential random variables with parameter β and the PDF

$$f(x) = \beta e^{-\beta x}, x \geq 0 \tag{1}$$

and the CDF

$$F(x) = 1 - e^{-\beta x}, x \geq 0 \tag{2}$$

and N be random variable with parameter λ and the probability density function

$$f_N(n) = \frac{e^{-\lambda}\lambda^n}{n!}, n \in \{0,1,2, \dots\} \tag{3}$$

In this article, we study a probability distribution of a new vector

$$(X, N) = \left(\bigwedge_{i=1}^{N} X_i, N\right), \tag{4}$$

where $\wedge$ denotes the minimum of the $\{X_i\}$. We shall denote to this distribution as the **JMEPM** distribution with parameter $\beta, \lambda > 0$, which stands for ***j***oint distribution of ***m***inimum of n IID

exponential random variables and ***P***oisson ***m***arginal. Note that the minimum of the n IID exponential variables $\{X_i\}$ has distribution with parameter β and PDF

$$f_{X/N=n}(x) = n\beta\left(e^{-\beta x}\right)^n, x \geq 0 \quad (5)$$

and CDF

$$F_{X/N=n}(x) = 1 - \left(e^{-\beta x}\right)^n, x \geq 0 \quad (6)$$

by using the formal of probability density of i-th order statistics $X_{(i)}, i = 1,2, \dots, n$, given below

$$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!\,(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x)$$

and CDF

$$F_{X_{(i)}}(x) = \sum_{i=1}^{n}\binom{n}{i}[F(x)]^{i-1}[1-F(x)]^{n-i}$$

(David and Nagaraja, 2003)

The **JMEPM** model has important applications in several fields of research including hydrology, climate, weather, and finance. The Poisson distribution is closely related to the exponential distribution in Poisson process where the waiting time between observations is exponential distributed and the number of observations is Poisson distribution (Ross, 2007), so that we study minimum customer wait time in this paper. (Park, 1970) who found properties of the joint distribution of the total time for the emission of particles from a radioactive substance but we discuss minimum time for the emission of particles. (Sarabia and Guillen, 2008) emphasize the value of studying a joint distribution, rather than Park's joint distribution with constant N. They use joint distribution of random vector (X, N). their application of interest is from the insurance industry where N represents the number of claims and X represents the total claim amount while we search minimum claim amount. (Al-Obaidi and Al-Khafaji, 2010) studies the joint distribution of this random vector (X, N) but X is a maximum of n IID exponential random variables.

Our paper is organized as follows. we present definitions and properties of the **JMEPM** distribution. It contains information about the marginals. Conditional distributions are offered in it. Moment and moment generated function are discussed in it. Estimation of parameters of this distribution are represented in it.

Definition and Basic Properties

Definition 1. A random vector (X, N) with stochastic representation(4), where $\{X_i\}$ are IID exponential variables(1)and N is a Poisson variable(3), independent of the $\{X_i\}$, is said to have a **JMEPM** distribution with parameter $\beta, \lambda > 0$, denoted by **JMEPM**(β, λ).

The joint distribution of (X, N) can be derived via hierarchical approach and standard technique of conditioning. Let $f(x, n)$ is joint PDF of (X, N) while $f_{X/N=n}(x) = f(x|N = n)$, then

$$f(x, n) = f_{X/N=n}(x)f_N(n) = f(x|N = n)f_N(n) \quad (7)$$

Theorem 1. The PDF of $(X, N)\sim$**JMEPM**$(\beta, \lambda)$is

$$f(x,n)=\begin{cases} n\beta e^{-n\beta x}\,\dfrac{e^{-\lambda}\lambda^{n}}{n!}, & x>0, n\in\{1,2,\ldots\} \\ e^{-\lambda} & x=n=0 \end{cases} \tag{8}$$

and CDF of $(X,N)\sim$**JMEPM**$(\beta,\lambda)$is

$$F(x,n)=\begin{cases} \displaystyle\sum_{n=1}^{[[N]]}\frac{e^{-\lambda}\lambda^{n}}{n!}\left(1-e^{-n\beta x}\right), & x>0, n\in\{1,2,\ldots\} \\ e^{-\lambda} & x=n=0 \end{cases} \tag{9}$$

Where $[[\cdot]]$ denotes the greatest integer function.

If $n=0$, then $f_N(n)=e^{-\lambda}$ and $f_{X/N=0}(x)=1$, so that $f(x,n)=e^{-\lambda}I_{(0,0)}$ because there are zero events. Thus we have

$$f(x,n)=e^{-\lambda}I_{\{(0,0)\}}+(1-e^{-\lambda})\frac{n\beta}{1-e^{-\lambda}}\left(e^{-\beta x}\right)^{n}\frac{e^{-\lambda}\lambda^{n}}{n!}I_{\mathbb{R}^{+}\times\mathbb{N}} \tag{10}$$

where I_A denotes the indicator function of the set A. We see that $f(x,n)$ has mixture distribution with probability $e^{-\lambda}$ is a point mass at {(0,0)}, or with probability $1-e^{-\lambda}$ is random vector $\left(\tilde{X},\tilde{N}\right)\in\mathbb{R}^{+}\times\mathbb{N}$ given by the PDF

$$g(x,n)=\frac{n\beta}{1-e^{-\lambda}}\left(e^{-\beta x}\right)^{n}\frac{e^{-\lambda}\lambda^{n}}{n!}, x>0, n\in\{1,2,\ldots\} \tag{11}$$

Proposition 1. If (X,N) has joint distribution $f(x,n)$, then

$$(X,N)\stackrel{d}{=}\left(I\tilde{X},I\tilde{N}\right) \tag{12}$$

Where $\left(\tilde{X},\tilde{N}\right)$is random vector with PDF $g(x,n)$given by(11) and I is an indicator random variable, independent of $\left(\tilde{X},\tilde{N}\right)$, taking on the values of 1 and 0 with probability $1-e^{-\lambda}$and $e^{-\lambda}$ respectively.

Marginal Distributions

It is useful to know univariate distributions for each variable. We will begin by finding the marginal distribution of $\tilde{X},\tilde{N}$ of the mixture representation in equation (12). We start with the marginal PDF and CDF of $\tilde{X}$.

Proposition 2. Let $\left(\tilde{X},\tilde{N}\right)$ has joint distribution $g(x,n)$, then the marginal probability distribution of $\tilde{X}$ is

$$f_{\tilde{X}}(x)=\frac{\beta\lambda e^{-\beta x}e^{-\lambda}}{1-e^{-\lambda}}\,e^{\lambda e^{-\beta x}}, x>0 \tag{13}$$

the marginal cumulative distribution function of $\tilde{X}$ is

$$F_{\tilde{X}}(x)=\frac{1}{1-e^{-\lambda}}\left(1-e^{-\lambda}e^{\lambda e^{-\beta x}}\right), x>0 \tag{14}$$

Next, we find the marginal CDF and PDF of X by using the equation(12). This implies $X\stackrel{d}{=}I\tilde{X}$, where $\tilde{X}$ is a continuous random variable with PDF and CDF in equations (13) and (14), respectively.

Proposition 3. Let $(X,N)\sim$**JMEPM**(β,λ), then the marginal cumulative distribution function of X is

$$F_X(x) = 1 + e^{-\lambda}\left(1 - e^{\lambda e^{-\beta x}}\right), x \geq 0 \tag{15}$$

and the marginal probability distribution of X is

$$f_X(x) = \begin{cases} \lambda\beta\, exp\left[\lambda e^{-\beta x} - \beta x - \lambda\right] & x > 0 \\ e^{-\lambda} & x = 0 \end{cases} \tag{16}$$

On the other hand, we find marginal PMF and CDF of $\widetilde{N}$ as the following.

Proposition 4. Let $(\tilde{X}, \widetilde{N})$ has joint distribution $g(x, n)$, then the marginal probability mass function of $\widetilde{N}$ is

$$f_{\widetilde{N}}(n) = \frac{e^{-\lambda}\lambda^n}{(1 - e^{-\lambda})n!} \tag{17}$$

the marginal cumulative distribution function of $\widetilde{N}$ is

$$F_{\widetilde{N}}(n) = \frac{e^{-\lambda}}{(1 - e^{-\lambda})}\sum_{i=1}^{n}\frac{\lambda^i}{i!} \tag{18}$$

By definition 1, the marginal PMF of N is Poisson distribution in equation (3). In view representation of Proposition 1, we have the relation $N = I\widetilde{N}$, where $\widetilde{N}$ has PMF in equation(17) while $I,$ independent of $\widetilde{N}$ takes on the values of 1 and 0 with probabilities $1 - e^{-\lambda}$ and $e^{-\lambda}$, respectively. Thus, it is clear that $N = 0$ if and only if $I = 0$, the probability of which is $e^{-\lambda}$, while for $n \in \mathbb{N}$, we have

$$P(N = n) = P\left(I\widetilde{N} = n\right) = \frac{e^{-\lambda}\lambda^n}{n!} \tag{19}$$

Conditional Distributions

Another important distribution is the conditional distribution, the probability of one variable when the value of the other variable is known. We saw in equation(10) that the conditional PDF of X for the case $N = 0$ is as follows:

$$f_{X/N=0}(x) = \begin{cases} 1 \;, & x \geq 0 \\ 0 \;, & x < 0 \end{cases}$$

Combining this with equation(5), we obtain

$$f_{X/N=n}(x) = \begin{cases} n\beta\left(e^{-\beta x}\right)^n & x \geq 0 \\ 1 & n = x = 0 \end{cases} \tag{20}$$

We can then note that the conditional CDF of X for the case $N = 0$ is

$$F_{X/N=0}(x) = \begin{cases} 1 \;, & x \geq 0 \\ 0 \;, & x < 0 \end{cases}$$

Combining this with equation(6), we obtain

$$F_{X/N=0}(x) = \begin{cases} 1 - \left(e^{-\beta x}\right)^n, & x \geq 0 \\ 1\,, & n = 0 \; and \; x \geq 0 \end{cases} \tag{21}$$

Next, we find the conditional PMF of N given $X = x$. We make the following proposition.

Proposition 5. Let $(X, N) \sim$ **JMEPM**(β, λ), then the conditional probability distribution of N given $X = x$ is

$$f_{N/X=x}(n) = \begin{cases} e^{-\lambda e^{-\beta x}} \dfrac{\left(\lambda e^{-\beta x}\right)^{n-1}}{(n-1)!}\,, & n \in \mathbb{N} \; and \; x > 0 \\ 1\,, & n = x = 0 \end{cases} \tag{22}$$

Note that $N/X = x$ is a shifted Poisson random variable. More precisely, for $x > 0$, $N/X = x$ has the same distribution as $Z + 1$, where Z is a Poisson with parameter $\lambda e^{-\beta x}$.

Moments and Moments generating function

We find various representations for bivariate and univariate moments connected with the joint distribution $f(x, n)$. To begin, we obtain a general expression for $E[X^\eta N^\gamma]$ and then proceed to obtain various special cases for particular values of η and γ. We recall in equation (12). This allows us to proceed as follows:

$$E[X^\eta N^\gamma] = E[I^{\eta+\gamma}] E\left[\tilde{X}^\eta \tilde{N}^\gamma\right] \tag{23}$$

We shall throughout that $\eta, \gamma \geq 0$. Noting that for $\eta + \gamma > 0$, we have

$$E[I^{\eta+\gamma}] = 1 - e^{-\lambda} \tag{24}$$

We obtain

$$E[X^\eta N^\gamma] = \begin{cases} E\left[\tilde{X}^\eta \tilde{N}^\gamma\right], & \eta = \gamma = 0 \\ \left(1 - e^{-\lambda}\right) E\left[\tilde{X}^\eta \tilde{N}^\gamma\right], & otherwise \end{cases} \tag{25}$$

Now, we state an expression for $E[X^\eta N^\gamma]$ in the proposition below.

Proposition 6. If $(X, N) \sim$ **JMEPM**(β, λ), then for any non-negative η and γ, we have

$$E[X^\eta N^\gamma] = \begin{cases} 1, & \eta = \gamma = 0 \\ \dfrac{\Gamma(\eta+1)}{\beta^\eta} \displaystyle\sum_{n=1}^{\infty} n^\gamma \frac{e^{-\lambda} \lambda^n}{n!\; n^\eta}\,, & otherwise \end{cases} \tag{26}$$

We can derive expressions for univarite moments by letting η or γ equal 0. The first moment in the univariate case represents the mean. This, in addition to the second moment, will reveal the variance. we will find the first and the second moments for X and N. Additionally, we will find the special case of $E[XN]$, which will be useful in deriving covariance of X and N.

A special case of $E[X^\eta N^\gamma]$ where $\gamma = 0$ gives us

$$E[X^\eta] = \begin{cases} 1, & \eta = 0 \\ \dfrac{\Gamma(\eta+1)}{\beta^\eta} \displaystyle\sum_{n=1}^{\infty} \frac{e^{-\lambda} \lambda^n}{n!\; n^\eta}\,, & otherwise \end{cases} \tag{27}$$

Therefore,

$$E[X] = \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!\ n}, E[X^2] = \frac{2}{\beta^2}\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!\ n^2}, E[XN] = \frac{1}{\beta}(1-e^{-\lambda})$$

Another special case of $E[X^\eta N^\gamma]$ where $\eta = 0$ gives us

$$E[N^\gamma] = \begin{cases} 1, & \gamma = 0 \\ \sum_{n=1}^{\infty} n^\gamma \frac{e^{-\lambda}\lambda^n}{n!}, & otherwise \end{cases} \tag{28}$$

As expected since N is a Poisson variable, then

$$E[N] = \lambda, E[N^2] = \lambda^2 + \lambda$$

The covariance matrix which has the following formal

$$\Sigma = \begin{bmatrix} \frac{e^{-2\lambda}}{\beta^2}\left(2\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!\ n^2} - \left[\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!\ n}\right]^2\right) & \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}\left(1-\frac{\lambda}{n}\right) \\ \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!}\left(1-\frac{\lambda}{n}\right) & \lambda \end{bmatrix} \tag{29}$$

We derive the moment generating function of joint PDF $f(x,n)$ in following proposition

Proposition 7. let (X,N)~**JMEPM**(β,λ), then the moment generating function of (X,N) is

$$M(t_1,t_2) = \sum_{n=1}^{\infty}\frac{e^{t_2 n}}{\left(1-\frac{t_1}{n\beta}\right)}\frac{e^{-\lambda}\lambda^n}{n!}, \qquad t_1,t_2 \in \mathbb{R} \tag{30}$$

Estimation

In practice the values of the parameters the joint distribution $f(x,n)$ are almost unknown and have to be estimated based on the sample data. That is why it is valuable to develop methods of estimation of our parameters, β and λ.

Moment estimators

Moment estimators are found by setting sample moments equal to the equation for the population moments and solving for the parameters which are to be estimated.

Proposition 8. Suppose $(X_1,N_1),(X_2,N_2),\ldots,(X_m,N_m)$ from a random sample where (X_i,N_i)~**JMEPM**(β,λ) and there exists at least one N_i such that $N_i > 0$. Then the moment estimators, $\tilde{\beta}$ and $\tilde{\lambda}$ of β and λ, respectively, are

$$\tilde{\beta} = \frac{1}{\bar{X}}\sum_{n=1}^{\infty}\frac{e^{-\lambda}\lambda^n}{n!\ n} \tag{31}$$

$$\tilde{\lambda} = \bar{N} \tag{32}$$

Where $\bar{X} = \sum_{i=1}^{m}\frac{X_i}{m}$ and $\bar{N} = \sum_{i=1}^{m}\frac{N_i}{m}$.

Maximum Likelihood Estimators

In this section we derive maximum likelihood estimators MLE for β and λ. We now state the MLE of β and λ in the preposition below.

Proposition 9. Suppose $(X_1, N_1), (X_2, N_2), \dots, (X_m, N_m)$ from a random sample where $(X_i, N_i) \sim$ **JMEPM**(β, λ) and there exists at least one N_i such that $N_i > 0$. Then the maximum likelihood estimators, $\hat{\beta}$ and $\hat{\lambda}$ of β and λ, respectively, are

$$\hat{\beta} = \frac{A}{\sum_{j\in C} N_j X_j} \tag{33}$$

$$\hat{\lambda} = \overline{N} \tag{34}$$

Where C =the set of all j's in the range $1,2,3,\dots,m$ such that $N_j > 0$, and $A = |C|$ = the number of elements in C.

CONCLUSIONS

We derived some properties of joint distribution of random vector (X, N), where N has Poisson distribution and X are minimum of N independent and identically distributed exponential random variables such as PDF and CDF of it. Also marginals and conditionals distributions of univariate random of this vector. Moments and moments generated function of the bivariate distribution and estimators of parameter of distribution of this random vector.

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