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A LINEAR OPERATOR OF A NEW CLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS

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ABSTRACT

In the present paper, we introduce a new class of meromorphic multivalent functions defined by linear derivative operator. We obtain some geometric properties, like, coefficient inequality, convex set, extreme points, distortion and covering theorem, δ-neighborhoods, partial sums and arithmetic mean.

Keywords: Meromorphic multivalent functions, Linear derivative operator, Extreme points, δ-neighborhoods, Partial sums.

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INTRODUCTION

Let M_p be the class of all functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p}, (p \in N = \{1,2, \dots\}, \qquad (1)$$

which are analytic and meromorphic multivalent in the punctured unit disk

$$U^* = \{z \in ₵ : 0 < |z| < 1\} = \mathrm{U}\backslash\{0\}.$$

Consider a subclass T_p of functions of the form:

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p}, \qquad (a_{k-p} \geq 0\,)\,. \qquad (2)$$

A function $f \in T_p$ is meromorphic multivalent starlike function of order $\rho(0 \le \rho < p)$ if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \rho, \qquad (0 \le \rho < p;\ z \in U^*). \qquad (3)$$

A functions $f \in T_p$ is meromorphic multivalent convex function of order $\rho(0 \le \rho < p)$ if

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \rho, \qquad (0 \le \rho < p;\ z \in U^*). \qquad (4)$$

The convolution (or Hadamard product) of two functions, f is given by (2) and

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}\, z^{k-p}, \ \left(b_{k-p} \ge 0\,, p \in N = \{1,2,\dots\}\right), \quad (5)$$

is defined by

$$(f * g)(z) = z^{-p} - \sum_{k=1}^{\infty} a_{k-p}\, b_{k-p} z^{k-p}.$$

We shall need to state the extended linear derivative operator of Ruscheweyh type for the function belonging to the class T_p which is defined by the following convolution

$$D_*^{\lambda,\mathrm{p}} f(z) = \frac{z^{-p}}{(1-z)^{\lambda+p}} * f(z), \ \left(\lambda > -p; f \in T_p\right). \qquad (6)$$

In terms of binomial coefficients, (6) can be written as

$$D_*^{\lambda,\mathrm{p}} f(z) = z^{-p} + \sum_{k=1}^{\infty} \binom{\lambda+k}{k} a_{k-p}\, z^{k-p}, \quad \left(\lambda > -p; f \in T_p\right) \qquad (7)$$

The linear operator $D^{\lambda,1}$ analogous to $D_*^{\lambda,1}$ was consider recently by Raina and Srivastava (2006) on the space of analytic and p-valent function in U $(U = U^* \cup \{0\})$.

Also the linear operator $D_*^{\lambda,\mathrm{p}}$ was studied on meromorphic multivalent functions for other class in (Goyal and Prajapat, 2009).

Definition 1: Let $f \in T_p$ be given by (2). The class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$ is defined by

$$E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)=\left\{f\in T_p:\left|\frac{z^{p+2}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)''+z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'-p^2}{\nu z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'+\alpha(1+\nu)p-p}\right|<\beta,\ \ (0\le\alpha<1,\right.$$

$$\left.0<\beta\le 1,\lambda>-p,0<\nu\le 1\,,p\in N\right\}. \qquad (8)$$

Najafzadeh and Ebadian (2013), Atshan and Kulkarni (2009), Atshan and Buti (2011), Khairnar and More (2008), studied meromorphic univalent and multivalent functions for different classes.

COEFFICIENT INEQUALITY

Theorem 1: Let $f \in T_p$. Then $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]a_{k-p}\le\beta p(1-\alpha)(1+\nu), \qquad (9)$$

$$(0\le\alpha<1,\qquad 0<\beta\le 1,\qquad \lambda>-p,\qquad 0<\nu\le 1,\qquad p\in N).$$

The result is sharp for the function

$$f(z)=z^{-p}+\frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}z^{k-p},\qquad k\ge 1.$$

Proof: Assume that the inequality (9) holds true and let $|z|$ =1, then from(8), we have

$$\left|z^{p+2}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)''+z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'-p^2\right|-\beta\left|\nu z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'+\alpha(1+\nu)p-p\right|$$

$$=\left|\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)^2a_{k-p}z^k\right|-\beta\left|p(1-\alpha)(1+\nu)-\nu\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)\,a_{k-p}z^k\right|$$

$$\le\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]a_{k-p}-\beta p(1-\alpha)(1+\nu)\le 0,$$

by hypothesis.

Hence, by the principle of maximum modulus, $f\in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$.
Conversely, suppose that f defined by (2) is in the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$.
Hence

$$\left|\frac{z^{p+2}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)''+z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'-p^2}{\nu z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'+\alpha(1+\nu)p-p}\right|$$

$$=\left|\frac{\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)^2 a_{k-p}z^k}{p(1-\alpha)(1+\nu)-\nu\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)\,a_{k-p}z^k}\right|<\beta\,,$$

Since $Re(z)<|z|$ for all z, we have

$$Re\left\{\frac{\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)^2 a_{k-p}z^k}{p(1-\alpha)(1+\nu)-\nu\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)\,a_{k-p}z^k}\right\}<\beta\,. \qquad (10)$$

We can choose the value of z on the real axis, so that $z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'$ is real. Let $z\to 1^-$, through real values, so we can write (10) as

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]a_{k-p}\le\beta p(1-\alpha)(1+\nu).$$

Finally sharpness follows if we take

$$f(z)=z^{-p}+\frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}z^{k-p}\,,\quad k\ge 1.$$

Corollary 1: Let $f\in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then

$$a_{k-p}\le\frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]},$$

where

$$(0\le\alpha<1,\quad 0<\beta\le 1,\quad \lambda>-p,\quad 0\le\nu\le 1\,,\quad p\in N).$$

CONVEX SET

Theorem 2: Let the functions

$$f(z)=z^{-p}+\sum_{k=1}^{\infty}a_{k-p}\,z^{k-p},\quad \left(a_{k-p}\ge 0\,\right),$$

$$g(z)=z^{-p}+\sum_{k=1}^{\infty}b_{k-p}\,z^{k-p},\quad \left(b_{k-p}\ge 0\,\right),$$

be in the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then for $0\le m\le 1$, the function

$$d(z)=\;(1-m)f(z)+\;m\,g(z)=z^{-p}+\sum_{k=1}^{\infty}c_{k-p}\,z^{k-p}, \qquad (11)$$

where

$$c_{k-p} = (1-m)a_{k-p} + mb_{k-p} \geq 0$$

is also in the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$.

Proof: Suppose that each of the functions f *and* g is in the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then, making use of Theorem 1, we see that

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]c_{k-p}$$

$$= (1-m)\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]a_{k-p} + m\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]b_{k-p}$$

$$\leq (1-m)\beta p(1-\alpha)(1+\nu) + m\beta p(1-\alpha)(1+\nu)$$

$$= \beta p(1-\alpha)(1+\nu),$$

which completes the proof of Theorem 2.

EXTREME POINTS

Theorem 3: Let $f_{-p} = z^{-p}$ and

$$f_{k-p}(z) = z^{-p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}z^{k-p}, \qquad (12)$$

for $k = 1,2,\ldots$. Then $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=0}^{\infty} d_{k-p}\, f_{k-p}(z),$$

where

$$d_{k-p} \geq 0 \quad \text{and} \quad \sum_{k=0}^{\infty} d_{k-p} = 1.$$

Proof: Suppose that

$$f(z) = \sum_{k=0}^{\infty} d_{k-p}\, f_{k-p}(z),$$

where

$$d_{k-p} \geq 0 \quad \text{and} \quad \sum_{k=0}^{\infty} d_{k-p} = 1.$$

Then

$$f(z) = d_{-p}f_{-p}(z) + \sum_{k=1}^{\infty} d_{k-p}\, f_{k-p}(z)$$

$$= d_{-p}z^{-p} + \sum_{k=1}^{\infty} d_{k-p}\left(z^{-p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}z^{k-p}\right)$$

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} \frac{\beta p(1-\alpha)(1+\nu)d_{k-p}}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}z^{k-p}$$

$$= z^{-p} + \sum_{k=1}^{\infty} Q_{k-p}z^{k-p},$$

where

$$Q_{k-p} = \frac{\beta p(1-\alpha)(1+\nu)d_{k-p}}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}.$$

By Theorem 1, we have $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$ if and only if

$$\sum_{k=1}^{\infty} \frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}Q_{k-p} \le 1,$$

for

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} Q_{k-p}z^{k-p}.$$

Hence

$$\sum_{k=1}^{\infty} \frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)} \times d_{k-p}\frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}$$

$$= \sum_{k=1}^{\infty} d_{k-p} = 1 - d_{-p} \le 1.$$

The proof is complete.

Conversely, assume $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then we show that f can be written in the form:

$$f(z) = \sum_{k=0}^{\infty} d_{k-p}\,f_{k-p}(z).$$

Now $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, implies from Theorem 1

$$a_{k-p} \le \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}.$$

Setting

$$d_{k-p} = \frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}a_{k-p}, \qquad k = 1,2,\ldots$$

and

$$d_{-p} = 1 - \sum_{k=1}^{\infty} d_{k-p},$$

then

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p}$$

$$= z^{-p} + \sum_{k=1}^{\infty} \frac{\beta p(1-\alpha)(1+\nu) d_{k-p}}{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}$$

$$= z^{-p} + \sum_{k=1}^{\infty} \left(f_{k-p} - z^{-p}\right) d_{k-p}$$

$$= z^{-p}\left(1 - \sum_{k=1}^{\infty} d_{k-p}\right) + \sum_{k=0}^{\infty} d_{k-p} f_{k-p}$$

$$= z^{-p} d_{-p} + \sum_{k=1}^{\infty} d_{k-p} f_{k-p}$$

$$= \sum_{k=0}^{\infty} d_{k-p}\, f_{k-p}(z).$$

DISTORTION AND COVERING THEOREM

Theorem 4: If the function $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, then for $0 < |z| < 1$

$$\frac{1}{|z|^p} - \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]} |z|^{1-p} \le |f(z)|$$
$$\le \frac{1}{|z|^p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]} |z|^{1-p}. \quad (13)$$

The result is sharp and attained for

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]} z^{1-p}.$$

Proof: Let $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then

$$|f(z)| = \left| \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p}\, z^{k-p} \right|$$

$$\le \frac{1}{|z|^p} + \sum_{k=1}^{\infty} a_{k-p}\, |z|^{k-p}$$

$$\leq \frac{1}{|z|^p} + |z|^{1-p} \sum_{k=1}^{\infty} a_{k-p} \,.$$

By Theorem1, we have

$$\sum_{k=1}^{\infty} a_{k-p} \leq \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]}\,.$$

Thus

$$|f(z)| \leq \frac{1}{|z|^p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]}|z|^{1-p}.$$

Similarly, we have

$$|f(z)| \geq \frac{1}{z^p} - \sum_{k=1}^{\infty} a_{k-p}\,|z|^{k-p}$$

$$\geq \frac{1}{z^p} - |z|^{1-p} \sum_{k=1}^{\infty} a_{k-p}$$

$$|f(z)| \geq \frac{1}{|z|^p} - \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]}|z|^{1-p}.$$

Hence result (13) follows.

Theorem 5: If $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, then for $0 < |z| < 1$

$$\frac{p}{|z|^{p+1}} - \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}[(1-p)+\beta\nu]}|z|^{-p} \leq |f'(z)| \leq \frac{p}{|z|^{p+1}} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}[(1-p)+\beta\nu]}|z|^{-p}\,, \quad (14)$$

with equality for

$$f(z) = \frac{1}{z^p} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}(1-p)[(1-p)+\beta\nu]} z^{1-p}.$$

Proof: Let $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$. Then

$$|f'(z)| \leq \frac{p}{|z|^{p+1}} + \sum_{k=1}^{\infty} (k-p) a_{k-p}\,|z|^{k-p-1}$$

$$\leq \frac{p}{|z|^{p+1}} + |z|^{-p} \sum_{k=1}^{\infty} (1-p) a_{k-p}$$

$$\leq \frac{p}{|z|^{p+1}} + \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}[(1-p)+\beta\nu]}|z|^{-p}\,.$$

On the other hand

$$|f'(z)| \geq \frac{p}{|z|^{p+1}} - \sum_{k=1}^{\infty}(k-p)a_{k-p}\,|z|^{k-p-1}$$

$$\geq \frac{p}{|z|^{p+1}} - |z|^{-p}\sum_{k=1}^{\infty}(1-p)a_{k-p}$$

$$\geq \frac{p}{|z|^{p+1}} - \frac{\beta p(1-\alpha)(1+\nu)}{\binom{\lambda+1}{1}[(1-p)+\beta\nu]}|z|^{-p},$$

which complete the proof.

NEIGHBORHOODS AND PARTIAL SUMS

Definition 2: Let $(0 \leq \alpha < 1, 0 < \beta \leq 1,\ \lambda > -p,\ 0 \leq \nu \leq 1, p \in N)$ *and* $\delta \geq 0$. We define the δ-neighborhood of a function $f \in T_p$ and denote $N_\delta(f)$ such that

$$N_\delta(f) = \left\{g \in T_p: g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p}\,z^{k-p}, and \sum_{k=1}^{\infty}\frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}\left|a_{k-p}-b_{k-p}\right| \leq \delta\right\}. \quad (15)$$

Goodman (1957), Ruscheweyh (1981) and Altintas and Owa (1996) have investigated neighborhoods for analytic univalent functions, we consider this concept for the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$.

Theorem 6: Let the function $f(z)$ defined by (2) be in the class $E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, for every complex number μ with $|\mu| < \delta,\ \delta \geq 0$,

let $\frac{f(z)+\mu z^{-p}}{1+\mu} \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, then $N_\delta(f) \subset E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta),\quad \delta \geq 0$.

Proof: Since $f \in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, f satisfies (9) and we can write for $\gamma \in \mathbb{C}, |\gamma| = 1$, that

$$\left[\frac{z^{p+2}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)'' + z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)' - p^2}{\nu z^{p+1}\left(D_*^{\lambda,\mathrm{p}}f(z)\right)' + \alpha(1+\nu)p - p}\right] \neq \gamma. \qquad (16)$$

Equivalently, we must have

$$\frac{(f*Q)(z)}{z^{-p}} \neq 0, \qquad z \in U^*, \qquad (17)$$

where

$$Q(z) = z^{-p} + \sum_{k=1}^{\infty} e_{k-p}\,z^{k-p},$$

such that

$$e_{k-p}=\frac{\gamma\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}$$

Satisfying

$$\left|e_{k-p}\right|\leq\frac{\gamma\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}\ and\ \ k\geq 1,\qquad p\in N.$$

Since

$$\frac{f(z)+\mu z^{-p}}{1+\mu}\in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta),$$

by (17)

$$\frac{1}{z^p}\left(\frac{f(z)+\mu z^{-p}}{1+\mu}*Q(z)\right)\neq 0. \qquad (18)$$

Now assume that $\left|\frac{(f*Q)(z)}{z^{-p}}\right|<\delta$. Then, by (18), we have

$$\left|\frac{1}{1+\mu}\frac{(f*Q)(z)}{z^{-p}}+\frac{\mu}{1+\mu}\right|\geq\frac{|\mu|}{|1+\mu|}-\frac{1}{|1+\mu|}\left|\frac{(f*Q)(z)}{z^{-p}}\right|>\frac{|\mu|-\delta}{|1+\mu|}\geq 0.$$

This is a contradiction as $|\mu|<\delta$. Therefore $\left|\frac{(f*Q)(z)}{z^{-p}}\right|\geq\delta$.

Letting

$$g(z)=z^{-p}+\sum_{k=1}^{\infty}b_{k-p}\,z^{k-p}\ \ \in N_\delta(f).$$

Then

$$\delta-\left|\frac{(g*Q)(z)}{z^{-p}}\right|\leq\left|\frac{(f-g)*Q(z)}{z^{-p}}\right|$$

$$\leq\left|\sum_{k=1}^{\infty}(a_{k-p}-b_{k-p})e_{k-p}z^{k-p}\right|$$

$$\leq\sum_{k=1}^{\infty}\left|a_{k-p}-b_{k-p}\right|\left|e_{k-p}\right||z|^{k-p}$$

$$<|z|^{k-p}\sum_{k=1}^{\infty}\left[\frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}\right]\left|a_{k-p}-b_{k-p}\right|$$

$$\leq\delta,$$

therefore $\frac{(g*Q)(z)}{z^{-p}}\neq 0$, and we get $g(z)\in E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$, so $N_\delta(f)\subset E^{\lambda,\mathrm{p}}(\nu,\alpha,\beta)$.

Theorem 7: Let $f(z)$ be defined by (2) and the partial sums $S_1(z)\ and\ \ S_q(z)$ be defined by

$S_1(z)=z^{-p}$ and

$$S_q(z) = z^{-p} + \sum_{k=1}^{q-1} a_{k-p}\, z^{k-p}\,, \qquad (q > 1).$$

Also suppose that

$$\sum_{k=1}^{\infty} C_{k-p}\, a_{k-p} \le 1\,,$$

where

$$C_{k-p} = \frac{\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]}{\beta p(1-\alpha)(1+\nu)}. \qquad (19)$$

Then we have

$$Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{C_q} \qquad (20)$$

$$Re\left\{\frac{S_q(z)}{f(z)}\right\} > 1 - \frac{C_q}{1+C_q}\,, \qquad (z \in U^*, \qquad q > 1). \qquad (21)$$

Each of the bounds in (19) and (20) is the best possible for $k \in N$.

Proof: For the coefficients C_{k-p} given by (19), it is not difficult to verify that $C_{k-p+1} > C_{k-p} > 1, \quad k = 1,2,\dots$.

Therefore, by using the hypothesis (19), we have

$$\sum_{k=1}^{q-1} a_{k-p} + C_q \sum_{k=q}^{\infty} a_{k-p} \le \sum_{k=1}^{\infty} C_{k-p} a_{k-p} \le 1. \qquad (22)$$

By setting

$$G_1(z) = C_q\left(\frac{f(z)}{S_q(z)} - \left(1 - \frac{1}{C_q}\right)\right)$$
$$= \frac{C_q \sum_{k=q}^{\infty} a_{k-p} z^k}{1 + \sum_{k=q}^{\infty} a_{k-p} z^k} + 1$$

and applying (22) we find that

$$\left|\frac{G_1(z)-1}{G_1(z)+1}\right| = \left|\frac{C_q \sum_{k=q}^{\infty} a_{k-p} z^k}{2 + 2\sum_{k=1}^{q-1} a_{k-p} z^k + C_q \sum_{k=q}^{\infty} a_{k-p} z^k}\right|$$

$$\le \frac{C_q \sum_{k=q}^{\infty} a_{k-p}}{2 - 2\sum_{k=1}^{q-1} a_{k-p} - C_q \sum_{k=q}^{\infty} a_{k-p}} \le 1.$$

This proof (20). Therefore, $Re(G_1(z)) > 0$ and we obtain

$$Re\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{C_q}.$$

Now, in the same manner, we can prove the assertion (21) by setting

$$G_2(z) = (1 + C_q)\left(\frac{S_q(z)}{f(z)} - \frac{C_q}{1+C_q}\right).$$

This completes the proof.

Theorem 8: Let $f_1(z), f_2(z), \dots, f_l(z)$ defined by

$$f_i(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p,i}\, z^{k-p}, \qquad \left(a_{k-p,i} \geq 0, \qquad i = 1,2,\dots,l, \quad k \geq 1\right) \qquad (23)$$

be in the class $E^{\lambda,p}(\nu,\alpha,\beta)$. Then the arithmetic mean of $f_i(z)$ $(i = 1,2,\dots,l)$ defined by

$$h(z) = \frac{1}{l}\sum_{i=1}^{l} f_i(z) \qquad (24)$$

is also in the class $E^{\lambda,p}(\nu,\alpha,\beta)$.

Proof: By (23), (24), we can write

$$h(z) = \frac{1}{l}\sum_{i=1}^{l}\left(z^{-p} + \sum_{k=1}^{\infty} a_{k-p,i}\, z^{k-p}\right)$$

$$= z^{-p} + \sum_{k=1}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l} a_{k-p,i}\right) z^{k-p}.$$

Since $f_i \in E^{\lambda,p}(\nu,\alpha,\beta)$ for every $(i = 1,2,\dots,l)$ so by using Theorem1, we prove that

$$\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]\left(\frac{1}{l}\sum_{i=1}^{l} a_{k-p,i}\right)$$

$$= \frac{1}{l}\sum_{i=1}^{l}\left(\sum_{k=1}^{\infty}\binom{\lambda+k}{k}(k-p)[(k-p)+\beta\nu]a_{k-p,i}\right)$$

$$\leq \frac{1}{l}\sum_{i=1}^{l}\beta p(1-\alpha)(1+\nu).$$

$$= \beta p(1-\alpha)(1+\nu).$$

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