Journal of Asian Scientific Research, 2013, 3(8):786-799

AESS

Journal of Asian Scientific Research

journal homepage: http://aessweb.com/journal-detail.php?id=5003

EXISTENCE AND UNIQUENESS LINEAR PARTIAL DIFFERENTIAL EQUATIONS DEPENDING INITIAL AND BOUNDARY CONDITIONS

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ABSTRACT

In this paper, we classify the linear second order partial differential equations. We will show that there are three types of partial differential equations hyperbolic, elliptic and parabolic. We are study hyperbolic equations and the type of equation will turn out to be decisive in establishing the kind of initial and boundary conditions that serve in a natural way to determine a solution uniquely.

Keywords: Hyperbolic equations, Characteristic method function, Partial differential equation, Canonical form.

1. INTRODUCTION

In (Kharibegashvili, 2008), the researcher considered one multidimensional version of the Cauchy characteristic problem in the light cone of the future for a hyperbolic equation with power nonlinearity with iterated wave operator in the principal part, depending on the exponent of nonlinearity and spatial dimension of equation, they are investigated the problem on the nonexistence of global solutions of the Cauchy characteristic problem Announce several new results concerning classical integration methods for second order scalar hyperbolic partial differential equations in plane, they are found that the vanishing of the generalized Laplace invariants is both necessary and sufficient for the equation to be Darbouxintegrable ,also invariantly characterize the various general cases of Darbouxintegrability due to Goursat, and researched necessary and sufficient conditions for an equation to admit a general or a complete intermediate integral in (MARTIN, 1996). In (Zigang, 2002), obtained necessary and sufficient conditions for the existence of diffeomorphisms that transform stochastic nonlinear systems to various canonical forms, this invariance rule allows the utilization of the existing necessary and sufficient conditions for deterministic nonlinear systems in associated stochastic nonlinear systems. A geometric setting for constrained exterior differential systems on fibered manifolds with n-dimensional bases is proposed, constraints given as sub manifolds of jet bundles locally defined by systems of first-order partial differential equations are shown to carry a natural geometric structure, called the canonical distribution. Systems of second-order partial differential equations subjected to differential constraints are modeled as exterior differential systems defined on constraint sub manifolds, as an important particular case, Lagrangian systems subjected to first-order differential constraints are considered in (Olga, 2006). A family of solutions of the Jacobi partial differential equations is investigated, this family is defined for arbitrary values of the dimension *n* of the Poisson system, it is also of an arbitrary nonlinearity and can be globally analyzed thus improving the usual local scope of the Darboux theorem in (Benito, 2009).In (Petrov, 2004), he suggested a new method for constructing asymptotic solutions of Hamiltonian systems of ordinary differential equations under the assumption that the Hamiltonian is a periodic functionof time and can be represented by a series in powers of a small parameter and presented algorithmsfor the solution of both the direct problem of constructing the asymptotic series for the phase flowmap on the basis of a given Hamiltonian and the inverse problem of constructing the Hamiltonian on the basis of a given phase flow map. The researcher presented an efficient approach for determining the solution of second-order linear differential equation, the second-order linear ordinary differential equation is first converted to a Volterra integral equation, by solving the resulting Volterra equation by means of Taylor's expansion, different approaches based on differentiation and integration methods are employed to reduce the resulting integral equation to a system of linear equation for the unknown and its derivatives the approximate solution of second-order linear differential equation is obtained in (Nadhem, 2013). In (Cemil, 2012), they are studied the boundedness of the solutions to a non-autonomous and non-linear differential equation of second order with two constant deviating arguments, also he extended some boundedness results obtained for a differential equation with a constant deviating argument in the literature to the boundedness of the solutions of a differential equation with two constant deviating arguments.

In (Haichun, 2011), the researcher considered the existence of the solution of the second-order impulsive differential equations with inconstant coefficients, and change the second-order impulsive partial differential equation into the equivalent equation by transformation, by using the critical point theory of variational method and Lax-Milgram theorem, they are obtained a new results for the existence of the solution of the impulsive partial differential equations. They studied the existence of global solutions for a class of second order impulsive abstract functional differential equations, the results are obtained by using Leray-Schauder's Alternative fixed point theorem in (Svasankaran *et al.*). In (Lahno and Zhdanov, 2005), the researchers perform complete group classification of the general class of quasilinear wave equations in two variables, this class may be seen as a generalization of the nonlinear d'Alembert, Liouville, sin/sinh-Gordon and Tzitzeica equations, they are derived a number of new genuinely nonlinear invariant models with high symmetry properties. In (Yan-Zhi Duan, 2009), he investigated the asymptotic behavior of classical solutions of reducible quasilinear hyperbolic systems with characteristic boundaries, under some suitable assumptions, and he proved that the solution approaches a combination of Lipchitz continuous and piecewise C^1 traveling wave solution. The hyperbolic system of plane ideal plasticity equations under the Saint_Venant_Mises' yield criterion is considered, its' characteristics curves are deformed by the action of admitted group of point transformations, that permits to construct a new analytical solution, the mechanical sense of obtained characteristic fields is discussed, the general algorithm of the relation of solutions of quasilinear hyperbolic system of two homogeneous equations of two independent variables is proposed in (Sergey *et al.*, 2009). In (Jeanne., 2008), they are presented a partial classification for C^{∞} type-changing symplectic Monge–Ampère partial differential equations that possess an infinite set of first-order intermediate partial differential equations.

The normal forms will be quasi-linear evolution equations whose types change from hyperbolic to either parabolic or to zero, the zero points can be viewed as analogous to singular points in ordinary differentia equations. The characteristic function method has been employed to determine and investigate certain classes of solution of a system of first-order nonlinear hydro dynamical equations of a perfect fluid with respect to different Coriolis parameters, the application of a one-parameter group of infinitesimal transformations reduces the number of independent variables by one, and consequently, the system of partial differential equations in two independent variables reduces to a system of ordinary differential equations, the resulting differential equations are solved analytically for some special cases in (Abd-el-Malek and Helal, 2005). In the present work, we consider the linear second order partial differential equations, and we study the three types of partial differential equations hyperbolic, elliptic and parabolic.

2. MATHEMATICAL CLASSIFICATION

We consider the following for linear partial differential equations:

$$\frac{\partial^2 u}{\partial t^2} - a^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.1}$$

$$\frac{\partial^2 u}{\partial t^2} - a^2(t)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.2}$$

$$a^2(t)\frac{\partial^2 u}{\partial t^2} - a^2(x)\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.3}$$

$$a^2(x)\frac{\partial^2 u}{\partial t^2} - a^2(t)\frac{\partial^2 u}{\partial x^2} = 0. \tag{2.4}$$

We can solve the equations (2.1), (2.2), (2.3) and (2.4) after classifying them as linear second order partial differential equations, recall that a linear second order partial differential equations in two variables is given by

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial t\,\partial x} + C\frac{\partial^2 u}{\partial x^2} + D\frac{\partial u}{\partial t} + E\frac{\partial u}{\partial x} + Fu = G, \tag{2.5}$$

where the all coefficients A, B, C, D, E, F are real functions of independent variables t, x . Define a discriminant $\Delta(t, x)$ by

$$\Delta(t_0, x_0) = B^2(t_0, x_0) - 4A(t_0, x_0)C(t_0, x_0). \tag{2.6}$$

Definition (1.1): An equation is called **hyperbolic** at the point (t_0, x_0) if $\Delta(t_0, x_0) > 0.$ It is **parabolic** at that point if $\Delta(t_0, x_0) = 0$ and **elliptic** if $\Delta(t_0, x_0) < 0$. The classification for partial differential equations which have more than two independent variables or higher order derivatives are more complicated.

The transformation leads to the discovery of special loci known as characteristic curves along which the partial differential equation provides only an incomplete expression for the second derivatives. Before we discuss the transformation to the canonical forms, we shall motivate the name and explain with more details why such transformation is useful. The name canonical form is used because this form corresponds to particularly simple choices of the coefficients of the second partial differential derivatives.

To transform the equation into a canonical form, we first show how a general transformation affect equation (2.5). Suppose ξ, η be twice continuously differentiable functions of t, x

$$\xi = \xi(t, x), \tag{2.7}$$

$$\eta = \eta(t, x). \tag{2.8}$$

Assume that the Jacobian J of the transformation defined by:

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial t} & \frac{\partial \xi}{\partial x} \\ \frac{\partial \eta}{\partial t} & \frac{\partial \eta}{\partial x} \end{vmatrix}, \tag{2.9}$$

is non-zero. This assumption is necessary to see that one can make the transformation back to the original variables t, x .

Use the chain rule to obtain all the partial derivatives required in (2.5). It is easy to ensure that

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t}, \tag{2.10}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}. \tag{2.11}$$

The second partial derivatives can be simplified to get:

$$\begin{aligned} \frac{\partial^2 u}{\partial t \partial x} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial t}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial t}\right) + \\ &\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial t}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial t \partial x} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial t \partial x}. \end{aligned} \tag{2.12}$$

In a similar method we get $\frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial t}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial t} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial t}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial t^2}, \tag{2.13}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2}. \tag{2.14}$$

Then putting these into (2.5) one finds after collecting like terms

$$A^* \frac{\partial^2 u}{\partial \xi^2} + B^* \frac{\partial^2 u}{\partial \xi \partial \eta} + C^* \frac{\partial^2 u}{\partial \eta^2} + D^* \frac{\partial u}{\partial \xi} + E^* \frac{\partial u}{\partial \eta} + F^* u = G^*, \tag{2.15}$$

where all the coefficients are now functions of ξ, η and

$$A^* = A\left(\frac{\partial \xi}{\partial t}\right)^2 + B\frac{\partial \xi}{\partial t}\frac{\partial \xi}{\partial x} + C\left(\frac{\partial \xi}{\partial x}\right)^2, \tag{2.16}$$

$$B^* = 2A\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial t} + B\left(\frac{\partial \xi}{\partial t}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial t}\right) + 2C\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}, \tag{2.17}$$

$$C^* = A\left(\frac{\partial \eta}{\partial t}\right)^2 + B\frac{\partial \eta}{\partial t}\frac{\partial \eta}{\partial x} + C\left(\frac{\partial \eta}{\partial x}\right)^2, \tag{2.18}$$

$$D^* = A\frac{\partial^2 \xi}{\partial t^2} + B\frac{\partial^2 \xi}{\partial t \partial x} + C\frac{\partial^2 \xi}{\partial x^2} + D\frac{\partial \xi}{\partial t} + E\frac{\partial \xi}{\partial x}, \tag{2.19}$$

$$E^* = A\frac{\partial^2 \eta}{\partial t^2} + B\frac{\partial^2 \eta}{\partial t \partial x} + C\frac{\partial^2 \eta}{\partial x^2} + D\frac{\partial \eta}{\partial t} + E\frac{\partial \eta}{\partial x}, \tag{2.20}$$

$$F^* = F, \tag{2.21}$$

$$G^* = G. \tag{2.22}$$

Obtaining equation (2.15) is in the same form as the original one. The classification of hyperbolic, parabolic and elliptic equations will not change under this transformation. The reason for this is that

$$\Delta^* = (B^*)^2 - 4A^*C^* = J^2(B^2 - 4AC) = J^2\Delta, \tag{2.23}$$

And since $J \neq 0$, the sign of Δ^* is the same as that of Δ .The classification depends only on the coefficients of the second derivative terms and thus we can write (2.5) and (2.15) respectively as:

$$A\frac{\partial^2 u}{\partial t^2} + B\frac{\partial^2 u}{\partial t \partial x} + C\frac{\partial^2 u}{\partial x^2} = H\left(t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right), \tag{2.24}$$

and

$$A^*\frac{\partial^2 u}{\partial \xi^2}+B^*\frac{\partial^2 u}{\partial \xi \partial \eta}+C^*\frac{\partial^2 u}{\partial \eta^2}=H^*\left(\xi,\eta,u,\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta}\right). \tag{2.25}$$

3. CANONICAL FORMS

In this section we discuss briefly the canonical forms, which correspond to particularly simple choices of the coefficients of the second partial derivatives of the unknown. To obtain a canonical form, we have to transform the partial differential equation which in turn will require the knowledge of characteristic curves. Three equivalent properties of characteristic curves, each of them can be used as a definition:

1. Initial data on a characteristic curve cannot be prescribed freely, but must satisfy a compatibility condition.
2. Discontinuities (of a certain nature) of a solution cannot occur except along characteristics.
3. Characteristics are the only possible “branch lines” of solutions, i.e. lines for which the same initial value problems may have several solutions.

Suppose we introduce specific choices for the functions ξ,η. This will be done in such a way that some of the coefficients $A^*, B^*,$ and C^* in (2.25) becomes zero.

3.1. Hyperbolic Equation

Note that A^*, B^* are similar and can be written as:

$$A\left(\frac{\partial \zeta}{\partial t}\right)^2+B\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x}+C\left(\frac{\partial \zeta}{\partial x}\right)^2. \tag{3.1.1}$$

In which ζ stands for either ξ or η. Let assume we try to choose ξ,η such that $A^*=C^*=0$. This is of course possible only if the equation is hyperbolic. (Recall that $\Delta^*=\left(B^*\right)^2-4A^*C^*$ and for this choice $\Delta^*=\left(B^*\right)^2>0$. Since the type does not change under the transformation, we must have a hyperbolic partial differential equation). In order to annihilate A^* and C^* we have to find ζ such that

$$A\left(\frac{\partial \zeta}{\partial t}\right)^2+B\frac{\partial \zeta}{\partial t}\frac{\partial \zeta}{\partial x}+C\left(\frac{\partial \zeta}{\partial x}\right)^2=0. \tag{3.1.2}$$

Dividing by $\left(\frac{\partial \zeta}{\partial x}\right)^2$, the above equation becomes:

$$A\left(\frac{\frac{\partial\zeta}{\partial t}}{\frac{\partial\zeta}{\partial x}}\right)^2 + B\left(\frac{\frac{\partial\zeta}{\partial t}}{\frac{\partial\zeta}{\partial x}}\right) + C = 0, \tag{3.1.3}$$

Thus the curve $\zeta(t,x) =$ constant. (3.1.4)

We have

$$d\zeta = \frac{\partial\zeta}{\partial t}dt + \frac{\partial\zeta}{\partial x}dx = 0, \tag{3.1.5}$$

Therefore,

$$\frac{\frac{\partial\zeta}{\partial t}}{\frac{\partial\zeta}{\partial x}} = -\frac{dx}{dt}, \tag{3.1.6}$$

And equation (3.1.3) becomes as follows:

$$A\left(\frac{dx}{dt}\right)^2 - B\frac{dx}{dt} + C = 0. \tag{3.1.7}$$

This is a quadratic equation for $\frac{dx}{dt}$ and its roots are:

$$\frac{dx}{dt} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{3.1.8}$$

These equations are called characteristic equations and are ordinary differential equations for families of curves in t, x plane along which $\zeta =$ constant. The solutions are called the characteristic curves. Notice that the discriminant is under the radical in (3.1.8) and since the problem is hyperbolic, $B^2 - 4AC > 0$, there are two distinct characteristic curves. We can choose one to be $\xi(t,x)$ and the other $\eta(t,x)$. Solving the ordinary differential equations (3.1.8), we obtain:

$$\phi_1(t,x) = C_1, \tag{3.1.9}$$

$$\phi_2(t,x) = C_2. \tag{3.1.10}$$

Thus the transformations:

$$\xi = \phi_1(t,x), \tag{3.1.11}$$

$$\eta = \phi_2(t,x), \tag{3.1.12}$$

They are will lead to $A^* = C^* = 0$ and the canonical form is:

$$B^{*}\frac{\partial^2 u}{\partial\xi\partial\eta}=H*, \tag{3.1.13}$$

Or after division by B^{*}

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\frac{H*}{B*}. \tag{3.1.14}$$

This is called the first canonical form of the hyperbolic equation.

Sometimes we find another canonical form for hyperbolic partial differential equations which is obtained by making a transformation

$$\alpha=\xi+\eta, \tag{3.1.15}$$

$$\beta=\xi-\eta. \tag{3.1.16}$$

Using the equations (3.1.6) ,(3.1.7) and (3.1.8) for this transformation one has

$$\frac{\partial^2 u}{\partial\alpha^2}-\frac{\partial^2 u}{\partial\beta^2}=H^{**}\left(\alpha,\beta,u,\frac{\partial u}{\partial\alpha},\frac{\partial u}{\partial\beta}\right). \tag{3.1.17}$$

Then the equation (3.1.17) is called the second canonical form of the hyperbolic equation.

3.2 Study Some Applications for Equations (3.1),(3.2),(2.3) And (2.4)

3.2.1. Let consider the problem (2.1) with $a^2(t)=t^2$ and

$$\frac{\partial^2 u}{\partial t^2}-t^2\frac{\partial^2 u}{\partial x^2}=0 \text{ for } x>0 .$$

Solution: $A=1, B=0, C=-t^2.$

If $\Delta=B^2-4AC$,

Then $\Delta=0-4\times1\times(-t^2)=4t^2.$

The equation is hyperbolic because $t>0$.

When $\frac{dx}{dt}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$

then the characteristic equation is : $\frac{dx}{dt}=\frac{0\pm\sqrt{4t^2}}{2\times1}=\frac{\pm2t}{2}=\pm t.$

These equations are separable ordinary differential equations and the solutions are:

$$x-\frac{1}{2}t^2=c_1,$$

$$x+\frac{1}{2}t^2=c_2.$$

We take then the following transformation:

$$\xi=x-\frac{1}{2}t^2. \tag{3.2.1.1}$$

$$\eta=x+\frac{1}{2}t^2. \tag{3.2.1.2}$$

Evaluate all derivatives of ξ,η for (3.2.1.1) and (3.2.1.2)

$$\frac{\partial\xi}{\partial t}=-t\ ,\frac{\partial^2\xi}{\partial t^2}=-1,\frac{\partial^2\xi}{\partial t\,\partial x}=0,\frac{\partial\xi}{\partial x}=1,\frac{\partial^2\xi}{\partial x^2}=0$$
$$\frac{\partial\eta}{\partial t}=t\ ,\frac{\partial^2\eta}{\partial t^2}=1,\frac{\partial^2\eta}{\partial t\,\partial x}=0,\frac{\partial\eta}{\partial x}=1,\frac{\partial^2\eta}{\partial x^2}=0.$$

Substituting all derivatives of ξ,η for (3.2.1.1) and (3.2.1.2) for $A^*,B^*,C^*,D^*,E^*,F^*,G^*$

$$A^*=0,B^*=-4t^2,C^*=0,D^*=-1,E^*=1,F^*=0,G^*=0.$$

Now solve (3.2.1.1) and (3.2.1.2) for t,x

$$x=\frac{\xi+\eta}{2},$$

$$t^2=\eta-\xi\Rightarrow t=\pm\sqrt{\eta-\xi},$$

And substituting in A^*,B^*,C^*,D^*,E^* we get

$$-4t^2\frac{\partial^2u}{\partial\xi\partial\eta}-\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}=0$$
$$\frac{\partial^2u}{\partial\xi\partial\eta}=4t^2+\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}=4(\eta-\xi)+\frac{\partial u}{\partial\xi}-\frac{\partial u}{\partial\eta}.$$

3.2.2. We solve the problem (2.2) with $a^2(x)=x^2$ and

$$\frac{\partial^2u}{\partial t^2}-x^2\frac{\partial^2u}{\partial x^2}=0 \text{ for } x>0.$$

Solution: $A=1,B=0,C=-x^2$

$\square = 0-4\times1\times\left(-x^2\right)=4x^2$

The equation is hyperbolic since $x>0$

Then the characteristic equation is : $\dfrac{dx}{dt}=\dfrac{0\pm\sqrt{4x^2}}{2\times1}=\dfrac{\pm2x}{2}=\pm x$

$$\frac{dx}{x}=\pm dt.$$

These equations are separable ordinary differential equations and the solutions are :

$$\ln x - t = c_1,$$
$$\ln x + t = c_2.$$

We take then the transformations:

$$\xi = \ln x - t, \tag{3.5.2.1}$$
$$\eta = \ln x + t. \tag{3.5.2.2}$$

Now , evaluate all derivatives of ξ,η for (3.5.2.1) and (3.5.2.2)

$$\frac{\partial\xi}{\partial t}=-1,\frac{\partial^2\xi}{\partial t^2}=0,\frac{\partial^2\xi}{\partial t\,\partial x}=0,\frac{\partial\xi}{\partial x}=\frac{1}{x},\frac{\partial^2\xi}{\partial x^2}=\frac{-1}{x^2}$$
$$\frac{\partial\eta}{\partial t}=1,\frac{\partial^2\eta}{\partial t^2}=0,\frac{\partial^2\eta}{\partial t\,\partial x}=0,\frac{\partial\eta}{\partial x}=\frac{1}{x},\frac{\partial^2\eta}{\partial x^2}=\frac{-1}{x^2}.$$

Substituting evaluate all derivatives of ξ,η for (3.5.2.1) and (3.5.2.2) for

$A^*,B^*,C^*,D^*,E^*,F^*,G^*.$

We get: $A^*=0,B^*=-4,C^*=0,D^*=1,E^*=1,F^*=0,G^*=0$

Then we solve (3.5.2.1) and (3.5.2.2) for t,x

$$t=\frac{\xi+\eta}{2},x^2=e^{\xi+\eta}\Rightarrow x=\pm\sqrt{e^{\xi+\eta}}$$

And substitute t,x in $A^*,B^*,C^*,D^*,E^*,F^*,G^*$, we get

$$-4\frac{\partial^2 u}{\partial\xi\partial\eta}+\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}=0$$
$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\frac{1}{4}\frac{\partial u}{\partial\xi}+\frac{1}{4}\frac{\partial u}{\partial\eta}$$

3.2.3. Consider the problem (2.3) with $a^2(t)=t^2$ and $a^2(x)=x^2$

$$t^2\frac{\partial^2 u}{\partial t^2}-x^2\frac{\partial^2 u}{\partial x^2}=0 \text{ for } t>0,x>0.$$

Solution: $A = t^2, B = 0, C = -x^2, \square = 4t^2x^2 > 0$

$$\frac{dx}{dt} = \pm \frac{x}{t},$$
$$\frac{dx}{x} = \pm \frac{dt}{t},$$

then $\ln x - \ln t = c_1, \ln x + \ln t = c_2.$

Take the following transformation:

$$\xi = \ln x - \ln t. \tag{3.5.3.1}$$
$$\eta = \ln x + \ln t. \tag{3.5.3.2}$$

Now, Evaluate all derivatives of ξ, η for equations (3.5.3.1) and (3.5.3.2)

$$\frac{\partial \xi}{\partial t} = -\frac{1}{t}, \frac{\partial \xi}{\partial x} = \frac{1}{x}, \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{t^2}, \frac{\partial^2 \xi}{\partial t \partial x} = 0, \frac{\partial^2 \xi}{\partial x^2} = -\frac{1}{x^2},$$
$$\frac{\partial \eta}{\partial t} = \frac{1}{t}, \frac{\partial \eta}{\partial x} = \frac{1}{x}, \frac{\partial^2 \eta}{\partial t^2} = -\frac{1}{t^2}, \frac{\partial^2 \eta}{\partial t \partial x} = 0, \frac{\partial^2 \eta}{\partial x^2} = -\frac{1}{x^2}.$$

Substitute all derivatives of ξ, η for equations (3.5.3.1) and (3.5.3.2) for

$A^*, B^*, C^*, D^*, E^*, F^*, G^*.$ We get

$$A^* = 0, B^* = -4, C^* = 0, D^* = 2, E^* = F^* = G^* = 0.$$

And solve (3.5.3.1) and (3.5.3.2) for t, x $t^2 = \frac{1}{e^{\xi - \eta}}, x^2 = e^{\xi + \eta}$.

Now substitute t^2, x^2 in $A^*, B^*, C^*, D^*, E^*, F^*, G^*$ we get:

$$-4 \frac{\partial^2 u}{\partial \xi \partial \eta} + 2 \frac{\partial u}{\partial \xi} = 0,$$
$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \frac{1}{2} \frac{\partial u}{\partial \xi}.$$

3.2.4. We can solve the problem if $a^2(x) = e^{2x}$ and $a^2(t) = t^2$

$$e^{2x} \frac{\partial^2 u}{\partial t^2} - t^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad t > 0, x > 0.$$

Solution: $A = e^{2x}, B = 0, C = -t^2, \square = 4e^{2x}t^2$

$$\frac{dx}{dt} = \pm\frac{t}{e^{x}},$$
$$e^{x} - \frac{1}{2}t^{2} = c_{1},$$
$$e^{x} + \frac{1}{2}t^{2} = c_{2}.$$

Take the following transformations:

$$\xi = e^{x} - \frac{1}{2}t^{2}. \qquad (3.5.4.1)$$

$$\eta = e^{x} + \frac{1}{2}t^{2}. \qquad (3.5.4.1)$$

Evaluate all derivatives ξ, η for (3.5.4.1) and (3.5.4.1), we get:

$$\frac{\partial\xi}{\partial t} = -t, \frac{\partial\xi}{\partial x} = e^{x}, \frac{\partial^{2}\xi}{\partial t^{2}} = -1, \frac{\partial^{2}\xi}{\partial t\,\partial x} = 0, \frac{\partial^{2}\xi}{\partial x^{2}} = e^{x},$$
$$\frac{\partial\eta}{\partial t} = t, \frac{\partial\eta}{\partial x} = e^{x}, \frac{\partial^{2}\eta}{\partial t^{2}} = 1, \frac{\partial^{2}\eta}{\partial t\,\partial x} = 0, \frac{\partial^{2}\eta}{\partial x^{2}} = e^{x}.$$

Substitute all derivatives of ξ, η for (3.5.4.1) and (3.5.4.1) for $A^{*}, B^{*}, C^{*}, D^{*}, E^{*}, F^{*}, G^{*}$. we obtain:

$$A^{*} = 0, B^{*} = -4t^{2}e^{2x}, C^{*} = 0, D^{*} = -e^{2x} - t^{2}e^{x}, E^{*} = e^{2x} - t^{2}e^{x}, F^{*} = 0, G^{*} = 0.$$

Then we can solve (3.5.4.1) and (3.5.4.2) for t, x $e^{x} = \frac{\xi+\eta}{2}, t^{2} = \eta - \xi \Rightarrow t = \pm\sqrt{\eta-\xi}$.

Now substitute t^{2}, x^{2} in $A^{*}, B^{*}, C^{*}, D^{*}, E^{*}, F^{*}, G^{*}$ we get:

$$\left(-4t^{2}e^{2x}\right)\frac{\partial^{2}u}{\partial\xi\partial\eta} - \left(e^{2x} + t^{2}e^{x}\right)\frac{\partial u}{\partial\xi} + \left(e^{2x} - t^{2}e^{x}\right)\frac{\partial u}{\partial\eta} = 0$$

$$\left(-4t^2e^{2x}\right)\frac{\partial^2 u}{\partial\xi\partial\eta}=\left(e^{2x}+t^2e^x\right)\frac{\partial u}{\partial\xi}-\left(e^{2x}-t^2e^x\right)\frac{\partial u}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\frac{\left(e^{2x}+t^2e^x\right)}{\left(-4t^2e^{2x}\right)}\frac{\partial u}{\partial\xi}+\frac{\left(e^{2x}-t^2e^x\right)}{\left(4t^2e^{2x}\right)}\frac{\partial u}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\left(\frac{1}{-4t^2}+\frac{1}{-4e^x}\right)\frac{\partial u}{\partial\xi}+\left(\frac{1}{4t^2}-\frac{1}{4e^x}\right)\frac{\partial u}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta}=\frac{1}{4}\left(\frac{1}{-t^2}+\frac{1}{-e^x}\right)\frac{\partial u}{\partial\xi}+\frac{1}{4}\left(\frac{1}{t^2}-\frac{1}{e^x}\right)\frac{\partial u}{\partial\eta}$$

4. CONCLUSION

The second order linear partial differential equations can be classified into three types (hyperbolic, elliptic and parabolic),which are invariant under changes of variables. The types are determined by sign of the discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. We concluded that hyperbolic equations have two distinct families of real characteristic curves. All the three types of two order partial differential equations can be converted to canonical forms after manipulated them by some steps. Hyperbolic equations converted to a form to be identical with the wave equation in the leading terms. Thus, the wave equation serves as canonical model for second order constant coefficient partial differential equations. We will spend the rest of the quarter studying the solution to the wave equation.

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