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ON A CERTAIN CLASS OF MULTIVALENT FUNCTIONS DEFINED BY HADAMARD PRODUCT

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ABSTRACT

The object of this paper to study the class $WH(n,p,\alpha,\lambda,\theta,q)$ *of multivalent functions defined by Hadamard product in the open unit disk* $U=\{z\in\mathbb{C}:|z|<1\}$. *We obtain some geometric properties for this class, like, coefficient estimate, closure theorem, extreme points, distortion theorem and modified Hadamard products.*

Keywords: Multivalent function, Hadamard product, Closure theorem, Distortion theorem, Extreme points, Modified hadamard products.

2000 Mathematics Subject Classification: 30C45, 30C50

1. INTRODUCTION

Let $W(p,n)$ denote the class of functions of the form:

$$f(z)=z^p-\sum_{k=n+p}^{\infty}a_kz^k\,(a_k\geq 0\,;k\geq n+p;p,n\in\mathbb{N}=\{1,2,\dots\}), \qquad (1.1)$$

which are analytic and multivalent in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$.
If $f\in W(p,n)$ is given by (1.1) and $g\in W(p,n)$ given by

$$g(z)=z^p-\sum_{k=n+p}^{\infty}b_kz^k\,, \qquad (b_k\geq 0\,)$$

then the Hadamard product (or convolution) $f*g$ of f and g is defined by

$$(f*g)(z)=z^p-\sum_{n=1}^{\infty}a_{n+p}b_{n+p}z^{n+p}=(g*f)(z). \qquad (1.2)$$

A function $f\in W(p,n)$ is said to be multivalent starlike of order α $(0\leq\alpha<p)$ if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)}\right\}>\alpha \quad (z\in U;\ 0\leq\alpha<p;p\in\mathbb{N}), \qquad (1.3)$$

and is said to be multivalent convex of order α $(0 \le \alpha < p)$ if it satisfies the condition:

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in U;\ 0 \le \alpha < p; p \in \mathbb{N}). \tag{1.4}$$

Denote by $S_n^*(p,\alpha)$ and $C_n(p,\alpha)$ the classes of multivalent starlike and multivalent convex functions of order α, respectively, which were introduced and studied by Owa (1992). It is known that (see (Goodman, 1983) and (Owa, 1992))

$$f \in C_n(p,\alpha) \text{ if and only if } \frac{zf'(z)}{p} \in S_n^*(p,\alpha). \tag{1.5}$$

The classes $S_1^*(p,\alpha) = S^*(p,\alpha)$ and $C_1(p,\alpha) = C(p,\alpha)$ were studied by Owa (1985).

Definition 1.1. Let f be given by (1.1), is said to be in the class $WH(n,p,\alpha,\lambda,\theta,q)$ if and only if satisfies the inequality:

$$Re\left\{\frac{z\big((f*g)(z)\big)^{(1+q)} + \lambda z^2\big((f*g)(z)\big)^{(2+q)}}{(1-\lambda)\big((f*g)(z)\big)^{(q)} + \lambda z\big((f*g)(z)\big)^{(1+q)}}\right\}$$

$$\ge \theta\left|\frac{z\big((f*g)(z)\big)^{(1+q)} + \lambda z^2\big((f*g)(z)\big)^{(2+q)}}{(1-\lambda)\big((f*g)(z)\big)^{(q)} + \lambda z\big((f*g)(z)\big)^{(1+q)}} - 1\right| + \alpha, \tag{1.6}$$

where $0 \le \alpha < p-q, p > q, n \in \mathbb{N}, q \in \mathbb{N}_0 = \{0,1,2,\ldots\}, 0 \le \lambda \le 1, \theta \ge 0$ and for each $f \in W(p,n)$, we have

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q}, \tag{1.7}$$

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j=0) \\ i(i-1)\ldots(i-j+1) & (j \ne 0) \end{cases}. \tag{1.8}$$

We note that by specializing the parameters $\lambda,\theta,q,n,p,$ we obtain the following different subclasses as studied by various authors:

(1) If $\theta = 0, \lambda = 0$, the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $TS_g^*(p,q,n,\alpha)$ which was studied by Aouf and Mostafa (2012).

(2) If $\theta = 0, \lambda = 1$, the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $TC_g(p,q,n,\alpha)$ which was studied by Aouf and Mostafa (2012).

(3) If $n = p = 1$ and $q = 0$, the family $WH(n,p,\alpha,\lambda,\theta,q)$ reduces to the class $WR(\lambda,\theta,\alpha)$ which was studied by Atshan and Buti (2011).

(4) If $\theta = 0, \lambda = 0, q = 0$ and replace $n+p$ by m, we have $WH(n,p,\alpha,0,0,0) = WH(p,m,\alpha)$ which was studied by Ali *et al.* (2006).

(5) If $\theta = 0, \lambda = 0, q = 0$ and $b_k = 1$ $(k \ge n+p)$, we have

$$WH(n,p,\alpha,0,0,0) = \begin{cases} T_n^*(p,\alpha) & \text{(Owa (1992))} \\ T_\alpha(p,\alpha) & \text{(Yamakawa (1992))} \end{cases}.$$

(6) $\theta = 0, \lambda = 1, q = 0$ and $b_k = 1$ $(k \ge n+p)$, we have

$$WH(n,p,\alpha,1,0,0) = \begin{cases} C_n(p,\alpha) & \text{(Owa (1992))} \\ C_\alpha(p,\alpha) & \text{(Yamakawa (1992))} \end{cases}.$$

(7) If $k = m, \theta = k, \alpha = \beta$ and $b_k = 1$, the family $WH(n, p, \alpha, \lambda, \theta, q)$ reduces to the class $k - UCV_p^n(\lambda, \beta, q)$ which was studied by Aqlan (2004).

Lemma 1 (Aqlan (2004) **).** Let $w = u + iv$. Then $Re(w) \geq \sigma$ if and only if $|w - (1 + \sigma)| \leq |w + (1 - \sigma)|$.

Lemma 2 (Aqlan (2004) **).** Let $w = u + iv$ and σ, η are real numbers. Then $Re(w) \geq \sigma|w - 1| + \eta$ if and only if $Re\{w(1 + \sigma e^{i\phi}) - \sigma e^{i\phi}\} > \eta$.

2. COEFFICIENT ESTIMATE

Theorem 1. Let the function f be in the form (1.1). Then f is in the class $WH(n, p, \alpha, \lambda, \theta, q)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{k!\,(1 + \lambda(k - q - 1))[(k - q)(1 + \theta) - (\theta + \alpha)]}{(k - q)!} a_k b_k$$

$$\leq \frac{p!\,(1 + \lambda(p - q - 1))(p - q - \alpha)}{(p - q)!}, \tag{2.1}$$

where $p, n \in \mathbb{N}, q \in \mathbb{N}_0, k \geq n + p, \theta \geq 0, 0 \leq \alpha < p - q, p > q,$ and $0 \leq \lambda \leq 1$.

The result is sharp for the function f given by

$$f(z) = z^p - \frac{p!(n+p-q)!(1+\lambda(p-q-1))(p-q-\alpha)}{(n+p)!(p-q)!(1+\lambda(n+p-q-1))[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}} z^{n+p}, \tag{2.2}$$

$(p, n \in \mathbb{N};\ p > q;\ q \in \mathbb{N}_0;\ z \in U)$.

Proof: Let $f \in WH(n, p, \alpha, \lambda, \theta, q)$. Then f satisfies the inequality (1.6) which is equivalent to

$$Re\left\{\frac{z((f * g)(z))^{(1+q)} + \lambda z^2((f * g)(z))^{(2+q)}}{(1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}}(1 + \theta e^{i\phi}) - \theta e^{i\phi}\right\}$$

$$\geq \alpha, \quad \text{(by using Lemma 2)}$$

$(0 \leq \alpha < p - q;\ p > q;\ \theta \geq 0;\ 0 \leq \lambda \leq 1;\ p \in \mathbb{N};\ q \in \mathbb{N}_0$ and $-\pi < \phi \leq \pi$. Or

$$Re\left\{\frac{\left[z((f * g)(z))^{(1+q)} + \lambda z^2((f * g)(z))^{(2+q)}\right](1 + \theta e^{i\phi})}{(1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}}\right.$$

$$\left. - \frac{\theta e^{i\phi}\left[(1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}\right]}{(1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}}\right\} \geq \alpha. \tag{2.3}$$

Let $g(z) = \left[z((f * g)(z))^{(1+q)} + \lambda z^2((f * g)(z))^{(2+q)}\right](1 + \theta e^{i\phi})$

$$-\theta e^{i\phi}\left[(1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}\right]$$

$h(z) = (1 - \lambda)((f * g)(z))^{(q)} + \lambda z((f * g)(z))^{(1+q)}$.

Then (2.3) is equivalent to

$|g(z) + (1 - \alpha)h(z)| \geq |g(z) - (1 + \alpha)h(z)|$ for $0 \leq \alpha < p - q$. (by using Lemma 1)

Now

$$|g(z)+(1-\alpha)h(z)| = \left\|\left[\frac{p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{k!\,a_k b_k}{(k-q-1)!}z^{k-q} + \frac{\lambda p!}{(p-q-2)!}z^{p-q}\right.\right.$$

$$\left. - \sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-2)!}z^{k-q}\right]\left(1+\theta e^{i\phi}\right) - \theta e^{i\phi}\left[\frac{(1-\lambda)p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\lambda)k!\,a_k b_k}{(k-q)!}z^{k-q}\right.$$

$$\left. + \frac{\lambda p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-1)!}z^{k-q}\right]$$

$$+ (1-\alpha)\left[\frac{(1-\lambda)p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{(1-\lambda)k!\,a_k b_k}{(k-q)!}z^{k-q}\right.$$

$$\left.\left. + \frac{\lambda p!}{(p-q-1)!}z^{p-q} - \sum_{k=n+p}^{\infty}\frac{\lambda k!\,a_k b_k}{(k-q-1)!}z^{k-q}\right]\right\|$$

$$= \left|\frac{p!}{(p-q)!}\left(1+\lambda(p-q-1)\right)(p-q+1-\alpha)z^{p-q}\right.$$

$$+ \frac{\theta e^{i\phi}p!}{(p-q)!}\left[(p-q)\left(1+\lambda(p-q-2)\right)-1+\lambda\right]z^{p-q}$$

$$- \sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\left(1+\lambda(k-q-1)\right)(k-q+1-\alpha)a_k b_k z^{k-q}$$

$$\left. - \sum_{k=n+p}^{\infty}\frac{\theta e^{i\phi}k!}{(k-q)!}\left[(k-q)\left(1+\lambda(k-q-2)\right)-1+\lambda\right]a_k b_k z^{k-q}\right|$$

$$\geq \frac{p!}{(p-q)!}\left(1+\lambda(p-q-1)\right)(p-q+1-\alpha)|z|^{p-q}$$

$$+ \frac{\theta p!}{(p-q)!}\left[(p-q)\left(1+\lambda(p-q-2)\right)-1+\lambda\right]|z|^{p-q}$$

$$- \sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\left(1+\lambda(k-q-1)\right)(k-q+1-\alpha)a_k b_k|z|^{k-q}$$

$$- \sum_{k=n+p}^{\infty}\frac{\theta k!}{(k-q)!}\left[(k-q)\left(1+\lambda(k-q-2)\right)-1+\lambda\right]a_k b_k|z|^{k-q}.$$

Similarly,

$$|g(z)-(1+\alpha)h(z)| = \left|\frac{p!}{(p-q)!}\left(1+\lambda(p-q-1)\right)(p-q-1-\alpha)z^{p-q}\right.$$

$$+ \frac{\theta e^{i\phi}p!}{(p-q)!}\left[(p-q)\left(1+\lambda(p-q-2)\right)-1+\lambda\right]z^{p-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q-1-\alpha)a_k b_k z^{k-q}$$

$$-\sum_{k=n+p}^{\infty}\frac{\theta e^{i\phi}k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k z^{k-q}\Bigg|$$

$$\le\frac{p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q-1-\alpha)|z|^{p-q}$$

$$+\frac{\theta p!}{(p-q)!}\big[(p-q)\big(1+\lambda(p-q-2)\big)-1+\lambda\big]|z|^{p-q}$$

$$+\sum_{k=n+p}^{\infty}\frac{k!}{(k-q)!}\big(1+\lambda(k-q-1)\big)(k-q-1-\alpha)a_k b_k|z|^{k-q}$$

$$+\sum_{k=n+p}^{\infty}\frac{\theta k!}{(k-q)!}\big[(k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\big]a_k b_k|z|^{k-q}.$$

Therefore

$$|g(z)+(1-\alpha)h(z)|-|g(z)-(1+\alpha)h(z)|\ge\frac{2p!}{(p-q)!}\big(1+\lambda(p-q-1)\big)(p-q-\alpha)$$

$$-\sum_{k=n+p}^{\infty}\frac{2k!}{(k-q)!}\Big[\big(1+\lambda(k-q-1)\big)(k-q-\alpha)$$

$$+\theta\Big((k-q)\big(1+\lambda(k-q-2)\big)-1+\lambda\Big)\Big]a_k b_k$$

$$\ge 0.$$

Hence

$$\sum_{k=n+p}^{\infty}\frac{k!\big(1+\lambda(k-q-1)\big)}{(k-q)!}[(k-q)(1+\theta)-(\theta+\alpha)]a_k b_k$$

$$\le\frac{p!\,(p-q-\alpha)}{(p-q)!}\big(1+\lambda(p-q-1)\big).$$

Conversely, by considering (2.1), we must show that

$$Re\left\{\frac{\Big[z\big((f*g)(z)\big)^{(1+q)}+\lambda z^2\big((f*g)(z)\big)^{(2+q)}\Big]\big(1+\theta e^{i\phi}\big)}{(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}}\right.$$

$$\left.-\frac{\big(\theta e^{i\phi}+\alpha\big)\Big[(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}\Big]}{(1-\lambda)\big((f*g)(z)\big)^{(q)}+\lambda z\big((f*g)(z)\big)^{(1+q)}}\right\}\ge 0. \qquad (2.4)$$

Upon choosing the values of z on the positive real axis where $0\le z=r<1, Re\big(-e^{i\phi}\big)\ge -\big|e^{i\phi}\big|=-1$ and letting $r\to 1^{-}$, we conclude to (2.4) by using (2.1) in the left hand of (2.2).

Corollary 1. Let f be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then

$$a_k \leq \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}, \tag{2.5}$$

where $p, n \in \mathbb{N}, q \in \mathbb{N}_0, 0 \leq \alpha < p-q, p > q, \theta \geq 0, k \geq n+p,$ and $0 \leq \lambda \leq 1$.

3. DISTORTION THEOREM

Theorem 2. Let the function $f \in WH(n,p,\alpha,\lambda,\theta,q)$. Then

$$\left[1-\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}$$
$$\leq \left|f^{(q)}(z)\right|$$
$$\leq \left[1+\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}.$$

The result is sharp for the function f given by (2.2).

Proof. Let $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$. Then

$$f^{(q)}(z) = \delta(p,q)z^{p-q} - \sum_{k=n+p}^{\infty} \delta(k,q)a_k z^{k-q},$$

where

$$\delta(i,j) = \frac{i!}{(i-j)!} = \begin{cases} 1 & (j=0) \\ i(i-1)\ldots(i-j+1) & (j \neq 0) \end{cases}.$$

Hence

$$f^{(q)}(z) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-q)!}a_k z^{k-q}.$$

By (2.5), we get

$$\left|f^{(q)}(z)\right|$$
$$\leq \frac{p!}{(p-q)!}|z|^{p-q}$$
$$+\frac{p!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{(p-q)!\,\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^{n+p-q}$$
$$= \left[1+\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q}$$

and

$$\left|f^{(q)}(z)\right|$$
$$\geq \left[1-\frac{\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}}|z|^n\right]\frac{p!}{(p-q)!}|z|^{p-q},$$

when $q = 0$, Theorem 2 would provide the growth property of functions in the class $WH(n,p,\alpha,\lambda,\theta,q)$. For $q \in \mathbb{N}$, the results may be looked upon as the distortion properties for the class $WH(n,p,\alpha,\lambda,\theta,q)$.

4. CLOSURE THEOREM

Let the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ be defined by

$$f_\iota(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,\iota} z^k \left(a_{k,\iota} \geq 0\right). \tag{4.1}$$

We shall prove the following result for the closure functions in the class $WH(n,p,\alpha,\lambda,\theta,q)$.

Theorem 3. Let the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function $h(z)$defined by

$$h(z) = \sum_{\iota=1}^{v} c_\iota f_\iota(z), \quad (c_\iota \geq 0), \tag{4.2}$$

is also in the $WH(n,p,\alpha,\lambda,\theta,q)$, where $\sum_{\iota=1}^{\infty} c_\iota = 1$.

Proof. According to the definition of $h(z)$, it can be written as

$$h(z) = \sum_{\iota=1}^{v} c_\iota \left(z^p - \sum_{k=n+p}^{\infty} a_{k,\iota} z^k\right) = \sum_{\iota=1}^{v} c_\iota z^p - \sum_{\iota=1}^{v}\sum_{k=n+p}^{\infty} c_\iota a_{k,\iota} z^k$$

$$= z^p - \sum_{k=n+p}^{\infty}\sum_{\iota=1}^{v} c_\iota a_{k,\iota} z^k. \tag{4.3}$$

Furthermore, since the functions $f_\iota(z)(\iota = 1,2,\dots,v)$ are in the class $WH(n,p,\alpha,\lambda,\theta,q)$, then

$$\sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} a_{k,\iota} b_k$$

$$\leq \frac{p!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(p-q)!}.$$

Hence

$$\sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} b_k \left(\sum_{\iota=1}^{v} c_\iota a_{k,\iota}\right)$$

$$= \sum_{\iota=1}^{v} c_\iota \left\{\sum_{k=n+p}^{\infty} \frac{k!\left(1+\lambda(k-q-1)\right)[(k-q)(1+\theta)-(\theta+\alpha)]}{(k-q)!} b_k a_{k,\iota}\right\}$$

$$\leq \frac{p!\left(1+\lambda(p-q-1)\right)(p-q-\alpha)}{(p-q)!},$$

which implies that $h(z)$be in the class $WH(n,p,\alpha,\lambda,\theta,q)$.

Corollary 2. Let the functions $f_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function $h(z)$defined by

$$h(z) = (1-t)\mathrm{f}_1(z) + t\mathrm{f}_2(z), \quad (0 \le t \le 1), \tag{4.4}$$

is also in the $WH(n,p,\alpha,\lambda,\theta,q)$.

5. EXTREME POINTS

We obtain here an extreme points of the class $WH(n,p,\alpha,\lambda,\theta,q)$.

Theorem 4. Let $\mathrm{f}_p(z) = z^p$ and

$$\mathrm{f}_k(z) = z^p - \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} z^k, \tag{5.1}$$

where $k \ge n+p, n,p \in \mathbb{N}, 0 \le \alpha < p-q, p > q, q \in \mathbb{N}_0, \theta \ge 0$ and $0 \le \lambda \le 1$.

Then the function f is in the class $WH(n,p,\alpha,\lambda,\theta,q)$ if and only if it can be expressed in the form:

$$\mathrm{f}(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k \mathrm{f}_k(z), \tag{5.2}$$

where $\big(\gamma_p \ge 0, \gamma_k \ge 0, k \ge n+p\big)$ and $\gamma_p + \sum_{k=n+p}^{\infty} \gamma_k = 1$.

Proof. Suppose that f is expressed in the form (5.2). Then

$$\mathrm{f}(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k \left[z^p - \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} z^k \right]$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} \gamma_k z^k .$$

Hence

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}$$

$$\times \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)\gamma_k}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}$$

$$= \sum_{k=n+p}^{\infty} \gamma_k = 1 - \gamma_p \le 1.$$

Then $f \in WH(n,p,\alpha,\lambda,\theta,q)$.

Conversely, suppose that $f \in WH(n,p,\alpha,\lambda,\theta,q)$. we may set

$$\gamma_k = \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_k,$$

where a_k is given by (2.4). Then

$$\mathrm{f}(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k} \gamma_k z^k$$

$$= z^p - \sum_{k=n+p}^{\infty} [z^p - \mathfrak{f}_k(z)]\,\gamma_k = \left(1 - \sum_{k=n+p}^{\infty} \gamma_k\right) z^p + \sum_{k=n+p}^{\infty} \gamma_k\, \mathfrak{f}_k(z) = \gamma_p z^p + \sum_{k=n+p}^{\infty} \gamma_k\, \mathfrak{f}_k(z).$$

This completes the proof of Theorem 4.

6. MODIFIED HADAMARD PRODUCT

Let the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ defined by (4.1). The modified Hadamard product of the functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ is defined by

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1} a_{k,2} z^k. \qquad (6.1)$$

Theorem 5. Let the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$and $b_{n+p} \geq b_k$ $(k \geq n+p)$. Then we have $(\mathfrak{f}_1 * \mathfrak{f}_2)(z) \in WH(n,p,\beta,\lambda,\theta,q)$, where $\beta = \frac{(p-q)(n+p)!(p-q)!\big(1+\lambda(n+p-q-1)\big)+p!(n+p-q)!\big(1+\lambda(p-q-1)\big)[\theta-(n+p-q)(1+\theta)]}{(n+p)!(p-q)!\big(1+\lambda(n+p-q-1)\big)-p!(n+p-q)!\big(1+\lambda(p-q-1)\big)}$.

(6.2)

The result is sharp for the functions $\mathfrak{f}_\iota(z)$ given by

$$\mathfrak{f}_\iota(z) = z^p - \frac{p!(n+p-q)!\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{(n+p)!(p-q)!\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]b_{n+p}} z^{n+p}, (\iota = 1,2).$$

(6.3)

Proof. Employing the technique used earlier by Schild and Silverman (1975), we need to find the largest $\beta = \beta(n,p,\alpha,\lambda,\theta,q)$ such that

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_{k,1} a_{k,2} \leq 1. \qquad (6.4)$$

Since the functions $\mathfrak{f}_\iota(z)$ $(\iota = 1,2)$ belong to the class $WH(n,p,\alpha,\lambda,\theta,q)$, then form Theorem 1,we have

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} a_{k,\iota} \leq 1. \qquad (6.5)$$

By the Cauchy Schwarz inequality, we have

$$\sum_{k=n+p}^{\infty} \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1. \qquad (6.6)$$

Thus, it is sufficient to show that

$$\frac{[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)}\sqrt{a_{k,1}a_{k,2}} \le \frac{[(k-q)(1+\theta)-(\theta+\alpha)]}{(p-q-\alpha)},$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{(p-q-\beta)[(k-q)(1+\theta)-(\theta+\alpha)]}{(p-q-\alpha)[(k-q)(1+\theta)-(\theta+\beta)]}. \qquad (6.7)$$

But from (6.6), we have

$$\sqrt{a_{k,1}a_{k,2}} \le \frac{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}, \qquad (6.8)$$

consequently, we need only to prove that

$$\frac{(p-q-\alpha)[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)[(k-q)(1+\theta)-(\theta+\alpha)]} \le \frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)},$$

or equivalently, that

$$\beta \le \frac{(p-q)k!(p-q)!\big(1+\lambda(k-q-1)\big)+p!(k-q)!\big(1+\lambda(p-q-1)\big)[\theta-(k-q)(1+\theta)]}{k!(p-q)!\big(1+\lambda(k-q-1)\big)-p!(k-q)!\big(1+\lambda(p-q-1)\big)}. \qquad (6.9)$$

Since the right hand side of (6.9) is an increasing function of k $(k \ge n+p)$.

Hence, we have

$$\beta = \frac{(p-q)(n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)+p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)[\theta-(n+p-q)(1+\theta)]}{(n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)-p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)}.$$

This completes the proof of Theorem 5.

Theorem 6. Let the functions $f_\iota(z)$ $(\iota = 1,2)$ defined by (4.1) be in the class $WH(n,p,\alpha,\lambda,\theta,q)$. Then the function

$$h(z) = z^p - \sum_{k=n+p}^{\infty}\left(a_{k,1}^2 + a_{k,2}^2\right)z^k, \qquad (6.10)$$

is in the class $WH(n,p,\beta,\lambda,\theta,q)$, where

$$\beta = \frac{(p-q)M + 2N\big(\theta-(n+p-q)(1+\theta)\big)}{M-2N}, \qquad (6.11)$$

such that

$$M = (n+p)!\,(p-q)!\,\big(1+\lambda(n+p-q-1)\big)[(n+p-q)(1+\theta)-(\theta+\alpha)]^2 b_{n+p},$$

and

$$N = p!\,(n+p-q)!\,\big(1+\lambda(p-q-1)\big)^2.$$

The result is sharp for the functions $f_\iota(z)(\iota = 1,2)$ given by (6.3).

Proof. From Theorem 1, we have

$$\sum_{k=n+p}^{\infty}\left\{\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}\right\}^2 a_{k,\iota}^2$$

$$\leq\left\{\sum_{k=n+p}^{\infty}\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}a_{k,\iota}\right\}^2$$

$$\leq 1,(\iota=1,2). \tag{6.12}$$

It follows that

$$\sum_{k=n+p}^{\infty}\frac{1}{2}\left\{\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)}\right\}^2\left(a_{k,1}^2+a_{k,2}^2\right)$$

$$\leq 1. \tag{6.13}$$

Therefore, we need to find the largest βsuch that

$$\frac{[(k-q)(1+\theta)-(\theta+\beta)]}{(p-q-\beta)}\leq\frac{1}{2}\frac{k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]^2b_k}{p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)(p-q-\alpha)^2},$$

that is, that

$$\beta\leq\frac{(p-q)M+2N\big(\theta-(k-q)(1+\theta)\big)}{M-2N}, \tag{6.14}$$

where

$$M=k!\,(p-q)!\,\big(1+\lambda(k-q-1)\big)[(k-q)(1+\theta)-(\theta+\alpha)]^2b_k,$$

and

$$N=p!\,(k-q)!\,\big(1+\lambda(p-q-1)\big)^2.$$

Since the right hand side of (6.14) is an increasing function of k and $b_{k+1}\geq b_k(k\geq n+p)$, then, setting $k=n+p$ in (6.14), we have (6.11).

This completes the proof of Theorem 6.

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