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ON A NEW CLASS OF UNIVALENT FUNCTIONS WITH APPLICATION OF FRACTIONAL CALCULUS OPERATORS DEFINED BY HOHLOV OPERATOR

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ABSTRACT

In the present paper , we discuss some properties of a new class of univalent functions in the unit disk defined by Hohlov operator with application of fractional calculus operators, like , distortion theorem using fractional calculus techniques for the class $H(a,b,c,\gamma,\beta)$,coefficient inequalities , extreme points , convex linear combination, and arithmetic mean. Also some results for our class are obtained.

Keywords: Univalent function, Fractional Calculus, Distortion theorem, Extreme points, Convex linear combinations and arithmetic mean.

Mathematics Subject Classification : 30C45.

1. INTRODUCTION

Let A denote the class of functions of the form:-

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1)$$

which are analytic and univalent in the open unit disk $U = \{z \in C: |z| < 1\}$.
If a function f is given by (1) and g is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad , \qquad (2)$$

is in the class A , the convolution(or Hadamard dproduct) of $f(z)$ and $g(z)$ is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad , z \in U. \qquad (3)$$

Let H denote the subclass of A consisting of functions of the form:-

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad , (a_n \geq 0) \qquad (4)$$

Definition(1)[1][5]:-The Gaussian hypergeometric function denoted by ${}_2F_1(a,b,c;z)$ and is defined by

$${}_2F_1(a,b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n \ , |z| < 1,$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$, $c > b > 0$ and $c > a + b$.

It is well known [2] that under the conditions $c>b>0$ and $c>a+b$, we have

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \ . \qquad (5)$$

Definition(2):- Let $f(z) \in H$ be of the form (4), then the Hohlov operator F(a,b,c),(F(a,b,c):H→H) [3] is defined by means of a Hadamard product below:

$$\mathrm{F(a,b,c)}f(\mathrm{z}) = \left(\mathrm{z}\,{}_2\mathrm{F}_1(\mathrm{a,b,c;z})\right) * f(\mathrm{z})$$

$$= z - \sum_{n=0}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n, \qquad (6)$$

$(a,b,c \in N, c \notin z_0^-, z \in U)$.

The integral representation of Hohlov operator is given by

$$F(a,b,c)f(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_0^1 \frac{(1-\sigma)^{c-a-b}\sigma^{b-2}}{\Gamma(c-a-b+1)}\ {}_2F_1(c-a, 1-a; c-a-b+1; 1-\sigma) f(z) d\sigma,$$

$(a>0, b>0, c-a-b+1>0, f \in H, z \in U)$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} I_{1,2}^{(a-2,b-2),(1-a,c-b)} f(z). \qquad (7)$$

Definition(3):-A function $f(z)$ in H is in the class H(a,b,c,γ,β,k) if and only if it satisfies the condition:-

$$\left|\frac{z(F(a,b,c)f(z))''' - \gamma(F(a,b,c)f(z))''}{\left(F(a,b,c)f(z)\right)'' + 2(1-\gamma)}\right| < \beta, \qquad (8)$$

where $0 \leq \gamma \leq 1, 0 < \beta \leq 1, z \in U$.

2.THE CLASS H(A,B,C,γ,β)

Theorem(1):-Let the function f be defined by (4).Then $f \in \mathrm{H}(\mathrm{a},\mathrm{b},\mathrm{c},\gamma,\beta)$if and only if

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\beta+\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n \leq 2\beta(1-\gamma), \qquad (9)$$

where $0 \leq \gamma \leq 1, 0 < \beta \leq 1$.

The result (9) is sharp for the function

$$f(z) = z - \frac{2\beta(1-\gamma)}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}} z^n, n \geq 2.$$

Proof:- Suppose that the inequality (9)holds true and |z|=1.Then we obtain

$$\left|z\left(F(a,b,c)f(z)\right)''' - \gamma\left(F(a,b,c)f(z)\right)''\right| - \beta\left|\left(F(a,b,c)f(z)\right)'' + 2(1-\gamma)\right|$$

$$= \left|-\sum_{n=2}^{\infty} n(n-1)(n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-2} + \gamma\sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-2}\right|$$

$$-\beta\left|2(1-\gamma) - \sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-2}\right|$$

$$\leq \sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n - 2\beta(1-\gamma)$$

$$\leq 0, \text{by hypothesis.}$$

Hence , by maximum modulus principle, $f \in H(a,b,c,\gamma,\beta)$.

Now, suppose that $f \in H(a,b,c,\gamma,\beta)$ so that

$$\left|\frac{z\big(F(a,b,c)f(z)\big)''' - \gamma\big(F(a,b,c)f(z)\big)''}{\big(F(a,b,c)f(z)\big)'' + 2(1-\gamma)}\right| < \beta, \quad z \in U,$$

then

$$\left|z(F(a,b,c)f(z))''' - \gamma\big(F(a,b,c)f(z)\big)''\right|$$

$$< \beta\left|\big(F(a,b,c)f(z)\big)'' + 2(1-\gamma)\right|,$$

we get

$$\left|-\sum_{n=2}^{\infty} n(n-1)(n-2)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-2}\right.$$

$$\left.+\ \gamma\sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-2}\right|$$

$$< \beta\left|2(1-\gamma) - \sum_{n=2}^{\infty} n(n-1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n z^{n-1}\right|,$$

thus

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\beta+\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n \le 2\beta(1-\gamma)\,.$$

Finally sharpness follows if we take

$$f(z) = z - \frac{2\beta(1-\gamma)}{n(n-1)(n-2+\beta+\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}z^n, n \ge 2.$$

The proof is complete.

Corollary (1):- Let $f \in H(a,b,c,\gamma,\beta)$.Then

$$a_n \le \frac{2\beta(1-\gamma)}{n(n-1)(n-2+\beta+\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}.$$

The result is sharp for the functions of the form:-

$$f(z) = z - \frac{2\beta(1-\gamma)}{n(n-1)(n-2+\beta+\gamma)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}z^n, n \ge 2.$$

3. APPLICATION OF THE FRACTIONAL CALCULUS

Various operators of fractional calculus (that is, fractional derivative and fractional integral)have been rather extensively studied by many researches [4-8].

However , we try to restrict ourselves to the following definitions given by Owa [9] for convenience .

Definition (4) (Fractional integral operator):- The fractional integral of order λ is defined for a function *f*(z), by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}}\, dt \quad ,(\lambda > 0), \qquad (10)$$

where *f*(z) is an analytic function in a simply –connected region of the z-plane containing the origin ,and the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real, when $z-t>0$.

Definition (5)(Fractional derivative operator):- The fractional derivatives of order λ, is defined, for a function *f*(z), by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}}\, dt \,, (\, 0 \le \lambda < 1), \qquad (11)$$

where *f*(z) is constrained, and the multiplicity of $(z-t)^{-\lambda}$ is removed , as in Definition(4) .

Definition(6):-Under the hypothesis of Definition (5) , the fractional derivative of order k+ λ is defined , for a function *f*(z) , by

$$D_z^{k+\lambda} f(z) = \frac{d^k}{dz^k} D_z^{\lambda} f(z), (0 \le \lambda < 1, k \in \mathrm{N}_0). \qquad (12)$$

Next, we state the following definition of fractional integral operator given by Srivastava, et al. [10].

Definition(7):- For real numbers $\alpha > 0, \eta \; and \; \delta$, the fractional operator , $I_{0,z}^{\alpha,\eta,\delta}$ is defined by

$$I_{0,z}^{\alpha,\eta,\delta} f(z) = \frac{z^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^z (z-t)^{\alpha-1} \mathrm{F}(\alpha+\eta, -\delta; \alpha; 1 - \frac{t}{z}) f(t) dt, \qquad (13)$$

where f(z) is analytic function in a simply connected region of the z-plane containing the origin with order

$f(z) = \mathrm{O}(|z|^{\epsilon})$, z→ 0, where $\epsilon > \max(0, \eta-\delta) - 1$,

$$F(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n}\, z^n$$

and $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n \in N \end{cases}$$

and the multiplicity of $(z-t)^{\alpha-1}$ is removed by requiring log(z-t) to be real,when z-t>0.

In order to prove our result concerning the fractional integral operator , we recall here the following lemma due to Srivastava, et al. [10].

Lemma(1):- Let $\alpha > 0$ and n>$\eta - \delta - 1$.Then

$$I_{0,z}^{\alpha,\eta,\delta} z^n = \frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)} z^{n-\eta}$$

Now making use of above Lemma 1 , we state and prove the following theorem:-

Theorem(2):- Let $\alpha > 0, \eta < 2, \alpha+\delta > -2, \eta(\alpha+\delta) \leq 3\alpha$.If f(z) defined by (4) is in the class H(a,b,c,γ, β), then

$$|I_{0,z}^{\alpha,\eta,\delta} f(z)| \geq \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)} \Big(1 - \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(\gamma+\beta)ab} |z|\Big) \quad (14)$$

and

$$|I_{0,z}^{\alpha,\eta,\delta} f(z)| \leq \frac{\Gamma(2-\eta+\delta)|z|^{1-\eta}}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)} \Big(1 + \frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(\gamma+\beta)ab} |z|\Big) \quad (15)$$

for z∈ U_0;where

$$U_0 = \begin{cases} U & n \leq 1 \\ U - \{0\} & n > 1 \end{cases}.$$

The result is sharp and is given by

$$f(z) = z - \frac{\beta(1-\gamma)}{ab(\gamma+\beta)} z^2. \quad (16)$$

Proof:-By using Lemma(1), we have

$$I_{0,z}^{\alpha,\eta,\delta} f(z) = \frac{\Gamma(2-\eta+\delta)}{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)} z^{1-\eta} - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n-\eta+\delta+1)}{\Gamma(n-\eta+1)\Gamma(n+\alpha+\delta+1)} a_n z^{n-\eta} \quad (17)$$

Setting

$$M(z)=\frac{\Gamma(2-\eta)\Gamma(2+\alpha+\delta)}{\Gamma(2-\eta+\delta)}z^{\eta}I_{0,z}^{\alpha,\eta,\delta}f(z)=z-\sum_{n=2}^{\infty}m(n)a_n z^n,$$

where

$$m(n)=\frac{(2-\eta+\delta)_{n-1}(1)_n}{(2-\eta)_{n-1}(2+\alpha+\delta)_{n-1}}\ ,(n\geq 2). \qquad (18)$$

It is easily verified that $m(n)$ is non-increasing for $n\geq 2$,and thus we have

$$0<m(n)\leq m(2)=\frac{2(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)}. \qquad (19)$$

Now, by application of Theorem (1) and (19), we obtain

$$|M(z)|\geq|z|-m(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\geq|z|-\frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(\gamma+\beta)ab}|z|^2,$$

which proves (14), and for (15) , we can find that

$$|M(z)|\leq|z|+m(2)|z|^2\sum_{n=2}^{\infty}a_n$$

$$\leq|z|+\frac{2c\beta(1-\gamma)(2-\eta+\delta)}{(2-\eta)(2+\alpha+\delta)(\gamma+\beta)ab}|z|^2,$$

and the proof is complete.

Taking $\eta=-\alpha=-\lambda$ and $\eta=-\alpha=\lambda$ in Theorem(2), we get two separate corollaries , which are contained in:-

Corollary(2):- Let the function f defined by (4) be in the class H(a,b,c,γ,β).Then we have

$$\left|D_z^{-\lambda}f(z)\right|\geq\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1-\frac{2c\beta(1-\gamma)}{(2+\lambda)(\gamma+\beta)ab}|z|\right), \qquad (20)$$

and

$$\left|D_z^{-\lambda}f(z)\right|\leq\frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)}\left(1+\frac{2c\beta(1-\gamma)}{(2+\lambda)(\gamma+\beta)ab}|z|\right). \qquad (21)$$

Corollary(3):- Let the function f defined by (4) be in the class H(a,b,c,γ,β).Then we have

$$\left|D_z^{\lambda} f(z)\right|$$

$$\geq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}\left(1-\frac{2c\beta(1-\gamma)}{(2-\lambda)(\gamma+\beta)ab}|z|\right), \qquad (22)$$

and

$$\left|D_z^{\lambda} f(z)\right|$$

$$\leq \frac{|z|^{1-\lambda}}{\Gamma(2-\lambda)}(1+\frac{2c\beta(1-\gamma)}{(2-\eta)(\gamma+\beta)ab}|z|). \qquad (23)$$

4. EXTREME POINTS

In the following theorem, we obtain extreme points for the class H(a,b,c,γ,β).

Theorem(3):-

Let $f_1(z)=z$ and $f_n(z)=z-\dfrac{2\beta(1-\gamma)}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}z^n$,

$n=2,3,4,\dots$.

Then $f \in$ H(a,b,c,γ,β) if and only if can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z),$$

where $\theta_k \geq 0\ and\ \sum_{n=1}^{\infty}\theta_k = 1$.In particular , the extreme points of H(a,b,c,γ,β) are the functions $f_1(z)=z\ \ and$

$$f_n(z)=z-\frac{2\beta(1-\gamma)}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}z^n, n=2,3,\dots .$$

Proof:- Firstly , let us express f as in the above theorem , therefore we can write

$$f(z)=\sum_{n=1}^{\infty}\theta_n f_n(z)$$

$$=\theta_1 z+\sum_{n=2}^{\infty}\theta_n[z-\frac{2\beta(1-\gamma)}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}z^n]$$

$$= z(\theta_1 + \sum_{n=2}^{\infty} \theta_n) - \sum_{n=2}^{\infty} \frac{2\beta(1-\gamma)\theta_n}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}} z^n$$

$$= z - \sum_{n=2}^{\infty} v_n z^n,$$

where

$$v_n = \frac{2\beta(1-\gamma)}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}} \theta_n .$$

Therefore, $f \in \mathrm{H}(a,b,c,\gamma,\beta)$, since

$$\sum_{n=2}^{\infty} \frac{v_n n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}{2\beta(1-\gamma)} = \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 < 1.$$

Conversely , assume that $f \in \mathrm{H}(a,b,c,\gamma,\beta)$.Then by (9), we may set

$$\theta_n = \frac{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}}{2\beta(1-\gamma)} a_n, n \geq 2 \; and \; 1 - \sum_{n=1}^{\infty} \theta_n = \theta_1.$$

Thus,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n = z - \sum_{n=2}^{\infty} \frac{2\beta(1-\gamma)\theta_n}{n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}} z^n$$

$$= z - \sum_{n=2}^{\infty} \theta_n\big(z - f_n(z)\big) = z(1 - \sum_{n=2}^{\infty} \theta_n) + \sum_{n=2}^{\infty} \theta_n f_n(z)$$

$$= \theta_1 z + \sum_{n=2}^{\infty} \theta_n f_n(z) = \sum_{n=1}^{\infty} \theta_n f_n(z).$$

This complete the proof.

5. DISTORTION THEOREM

In the following theorem , we obtain the distortion bounds for $f \in \mathrm{H}(a,b,c,\gamma,\beta)$.

Theorem(4):- Let $f \in \mathrm{H}(a,b,c,\gamma,\beta)$.Then

$$r-\frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}r^2 \le |f(z)| \le r+\frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}r^2, |z|=r < 1. \qquad (24)$$

The result is sharp for the function

$$f(z) = z-\frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}z^2. \qquad (25)$$

Proof:- By using (9) and corollary (1) , we obtain

$$2(\gamma+\beta)\frac{ab}{c}\sum_{n=2}^{\infty}a_n \le \sum_{n=2}^{\infty}n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_n \le 2\beta(1-\gamma).$$

This implies that

$$\sum_{n=2}^{\infty}a_n \le \frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}. \qquad (26)$$

For the function $f(z)=z-\sum_{n=2}^{\infty}a_n z^n$ and using (26) and $|z|=r<1$, *we have*

$$|f(z)| \le |z|+\sum_{n=2}^{\infty}a_n|z|^n \le r(1+r\sum_{n=2}^{\infty}a_n) \le r+\frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}\,r^2.$$

Similarly

$$|f(z)| \ge |z|-\sum_{n=2}^{\infty}a_n|z|^n \ge r(1-r\sum_{n=2}^{\infty}a_n) \ge r-\frac{c\beta(1-\gamma)}{(\gamma+\beta)ab}\,r^2.$$

Hence the proof is complete.

Corollary(4):- Let $f \in \mathrm{H}(a,b,c,\gamma,\beta)$. Then

$$1-\frac{2c\beta(1-\gamma)}{(\gamma+\beta)ab}\,r \le |f'(z)| \le 1+\frac{2c\beta(1-\gamma)}{(\gamma+\beta)ab}\,r. \qquad (27)$$

The result is sharp for the function given by (25).

6. CONVEX LINEAR COMBINATION

In the following theorem , we show that this class $\mathrm{H}(a,b,c,\gamma,\beta)$ is closed under convex linear combination.

Theorem(5):-The class $\mathrm{H}(a,b,c,\gamma,\beta)$ is closed under convex linear combination.

Proof:- We want to show the function

$$\mathrm{K(z)} = (1-\mu)f_1(z) + \mu f_2(z), 0 \le \mu \le 1 \qquad (28)$$

is in the class $\mathrm{H}(a,b,c,\gamma,\beta)$,where $f_1(z), f_2(z) \in \mathrm{H}(a,b,c,\gamma,\beta)$ and

$$f_1(z) = z + \sum_{n=2}^{\infty} a_{n,1}\, z^n, f_2(z) = z + \sum_{n=2}^{\infty} a_{n,2}\, z^n.$$

By (9) , we have

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_{n,1} \le 2\beta(1-\gamma)$$

and

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_{n,2} \le 2\beta(1-\gamma), \qquad (29)$$

Therefore

$$\mathrm{K(z)} = (1-\mu)f_1(z) + \mu f_2(z)$$

$$= (1-\mu)(z + \sum_{n=2}^{\infty} a_{n,1}\, z^n) + \mu(z + \sum_{n=2}^{\infty} a_{n,2}\, z^n)$$

$$= z + \sum_{n=2}^{\infty} [(1-\mu)a_{n,1} + \mu a_{n,2}]z^n.$$

We must show K(z) with the coefficient $((1-\mu)a_{n,1} + \mu a_{n,2})$ satisfy in the relation (9) also the coefficient $((1-\mu)a_{n,1} + \mu a_{n,2})$ satisfy in the inequality in corollary(1).Further ,

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}((1-\mu)a_{n,1} + \mu a_{n,2})$$

$$= \sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}(1-\mu)a_{n,1}$$

$$+ \sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\mu a_{n,2}$$

$$\le (1-\lambda)2\beta(1-\gamma) + \lambda 2\beta(1-\gamma) = 2\beta(1-\gamma).$$

Therefore, it follows that K(z) is in the class $\mathrm{H}(a,b,c,\gamma,\beta)$.

7. ARITHMETIC MEAN

In the following theorem , we shall prove that the class $\mathrm{H}(a,b,c,\gamma,\beta)$ is closed under arithmetic mean.

Theorem(6):-Let $f_1(z)$, $f_2(z)$ .... $f_l(z)$ defined by

$$f_i(z) = z+\sum_{n=2}^{\infty} a_{n,i}\, z^n, \left(a_{n,i}\geq 0, i=1,2,\ldots l, n\geq 2\right) \qquad (30)$$

be in the class $\mathrm{H}(a,b,c,\gamma,\beta)$. Then the arithmetic mean of $f_i(z)$ $(i=1,2,\ldots,l)$ defined by

$$h(z) = \frac{1}{l}\sum_{i=1}^{l} f_i(z), \qquad (31)$$

is also in the class $\mathrm{H}(a,b,c,\gamma,\beta)$.

Proof:- By (30),(31) we can write

$$h(z)=\frac{1}{l}\sum_{i=1}^{l}(z+\sum_{n=2}^{\infty}a_{n,i}\,z^n)=z+\sum_{n=2}^{\infty}(\frac{1}{l}\sum_{i=1}^{l}a_{n,i})z^n.$$

Since $f_i(z)\in \mathrm{H}(a,b,c,\gamma,\beta)$ for every i=1,2,…,l , so by using Theorem(1) , we prove that

$$\sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}(\frac{1}{l}\sum_{i=1}^{l}a_{n,i})$$

$$=\frac{1}{l}\sum_{i=1}^{l}(\sum_{n=2}^{\infty} n(n-1)(n-2+\gamma+\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}a_{n,i})\leq\frac{1}{l}\sum_{i=1}^{l}2\beta(1-\gamma).$$

The proof is complete.

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