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WHEN A FULLY D-STABLE MODULE IS QUASI-PROJECTIVE

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ABSTRACT

The concept of full d-stability of modules was introduced by the authors in a previous paper. It is stronger than the duo property. Although quasi-projectivity is not a necessary condition for full d-stability, it is inherent with this property in most of the known examples in such a way that it was difficult to give an example of a fully d-stable module which is not quasi-projective. This motivates the subject of this paper.

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1. INTRODUCTION

In a previous paper [1], we introduce the concept of full d-stability of modules . A submodule N of an R-module M is said to be d-stable if $N \subset Ker(\alpha)$ for every homomorphism $\alpha : M \to M/N$, the module M is said to be fully d-stable, if each of its submodules is d-stable [1]. Full d-stability is stronger than the duo property, the Z- module $Z_{(p^\infty)}$ is a duo module which is not fully d-stable, while any fully d-stable module is duo [1]. Any quasi-projective duo module is fully d-stable [1], examples of fully d-stable modules, which are not quasi-projective, exist (see [1] and [2]), but it seems that quasi-projectivity inherent most of the known fully d-stable modules. Any commutative ring (with identity) is fully d-stable module over itself and it is quasi-projective (since it is projective); likewise, any cyclic module over commutative ring; the Z- modules Z/nZ; the Z- module $\oplus_{p\in PR} Z/pZ$ [1]. In this paper the conditions and the cases, under which, full d-stability implies quasi-projectivity, will be investigated and certain examples will be studied .

All rings considered throughout are with identity, and modules are left and unitary.

2. RESULTS

We start with known concepts about modules and relations between those concepts.

Definition-2.1.

Abbas and Asaad [1] An R-module M is said to be *fully d-stable*, if $\mathrm{N} \subset \ker \alpha$ for each submodule N of M and each R-homomorphism $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$.

Definition-2.2.

Ozcan, et al. [3] An R-module M is said to be *duo*, if every submodule of M is fully invariant.

Proposition-2.3.

Abbas and Asaad [1] Any fully d-stable module is duo.

Definition-2.4.

Fuchs and Rangaswamy [4] An R-module M is said to be *quasi-projective*, if for each submodule N of M and each R-homomorphism $\alpha : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ there exists an R-endomorphism f of M such that $\pi \circ f = \alpha$ where $\pi : \mathrm{M} \to \mathrm{M}/\mathrm{N}$ is the natural epimorphism.

Proposition-2.5.

Abbas and Asaad [1] Any quasi-projective duo module is fully d-stable.

Multiplication modules were introduced by A. Barnard in 1981 (see [5]), as a generalization to multiplication ideals. A left R-module M is called *multiplication module* if every submodule N of M is of the form IM, for some ideal I of R [5]. The ring R was assumed commutative in the definition of multiplication module by Barnard (and by many other authors that studied this concept later), see for example([6], [7], [8], [9], [10]). Smith [9] proved that, in the class of projective modules (over commutative ring), the two concepts, duo and multiplication are equivalent, that is, the intersection of the class of duo modules and the class of projective modules is precisely the class of multiplication modules. While, in general, multiplication modules is a subclass of duo modules. This fact is true even if the ring was not commutative. Tuganbaev [11] investigated multiplication modules over rings which are not necessary commutative, and generalized most of the results that were deduced for multiplication modules over commutative rings to that over certain noncommutative rings or even arbitrary rings see [11] . In this work, particularly in investigating the relation with fully d-stable modules, multiplication modules are assumed over rings not necessarily commutative (Tuganbaev version), unless stated.

Definition-2.6.

Tuganbaev [11] Let R be any ring. An R-module M is said to be *multiplication*, if for any submodule N of M there is an ideal I of R such that $\mathrm{N}=\mathrm{IM}$.

In the following we will see the relation between multiplication and full d-stability.

Proposition-2.7.

Let R be any ring. If M is a multiplication R-module, then it is fully d-stable.

Proof: Let N be a submodule of a multiplication R-module M, then $\mathrm{N}=\mathrm{IM}$, for some ideal I of R. If $\alpha:\mathrm{M}\rightarrow\mathrm{M/N}$ is an R-homomorphism, then $\alpha(\mathrm{N})=\alpha(\mathrm{IM})=\mathrm{I}\alpha(\mathrm{M})\subset\mathrm{I(M/N)}=\mathrm{IM/N}=\mathrm{N/N}$, that is, $\mathrm{N}\subset\ker\alpha$. ◊

The converse of proposition (2.7) is not true in general, an example will be given , but we need to recall the following two facts.

Theorem-2.8.

Smith [10] Let $\mathrm{M}_{\lambda}(\lambda\in\Lambda)$ be a collection of finitely generated R-modules and $\mathrm{M}=\oplus_{\lambda\in\Lambda}\mathrm{M}_{\lambda}$. Then M is a multiplication module if and only if M_{λ} is a multiplication module and ann(M_{λ}) + ann($\hat{\mathrm{M}}_{\lambda}$) = R for each $\lambda\in\Lambda$.

(where the notation $\hat{\mathrm{M}}_{\lambda}$ means the direct sum of all M_{μ}, $\mu\in\Lambda$ *and* $\mu\neq\lambda$). ◊

Theorem-2.9.

Fuchs and Rangaswamy [4] An abelian group A is quasi-projective if and only if, it is:

1. free, or
2. a torsion group such that every p-component A_p is a direct sum of cyclic groups of the same order p^n. ◊

It is clear that the above theorem is true, if "abelian group" is replaced by " Z-module"**.**

Example-2.10.

Let $\mathrm{M}=\oplus\mathrm{M}_p$, p runs over all prim numbers, where $\mathrm{M}_p=Z/pZ$, then M is a Z-module which is not multiplication by theorem (1.8) (note that each M_p is multiplication but, for fixed p**,** ann(M_p) + ann($\hat{M}_p$) = pZ**).** On the other hand M is quasi-projective by Theorem (2.9, 2.) also it is duo(see [3]), hence by Proposition 2.5 it is fully d-stable. ◊

Remarks-2.11.

The relationships deduced above can be summarized as below:

i) Over any ring : full d-stability $\Rightarrow$ duo

The arrow is not reversed in general but it does with quasi-projectivity.

$Z_{(p^{\infty})}$ is an example of a duo Z-module which is not fully d-stable (see [1]). Also $Z \oplus Z$ is a quasi-projective Z-module which is not duo. Two examples in Abbas and Asaad [1] and Abbas and Asaad [2] were given of fully d- stable modules which are not quasi-projective.

ii) Over any ring: multiplication $\Rightarrow$ full d-stability $\Rightarrow$ duo.

For projective modules (over commutative ring), multiplication and duo properties are equivalent (see [9], corollary B). This fact and propositions (2.3), (2.5) lead to the following:

iii) For projective modules over commutative ring:

multiplication $\Leftrightarrow$ full d-stability $\Leftrightarrow$ duo

Let R be a principal ideal domain. It is known that a finitely generated torsion R-module is a direct sum of cyclic modules each of which isomorphic to a quotient R/I, for some ideal I of R (see for example [4]), (note that R/I is quasi-projective for each I (see [4]). On the other hand, "if a module M is a direct sum of quasi-projective modules and M is duo, then M is quasi-projective too([4]). Using these facts we have the following result.

Proposition-2.12.

Let R be a principal ideal domain, and M a finitely generated torsion R-module. If M is fully d-stable (multiplication), then it is quasi-projective.

Proof. Since multiplication implies full d-stability, it is enough to prove the proposition when M is fully d-stable. By the above notes, M is a direct sum of quasi-projective modules, and full d-stability implies that M is duo, hence by the above note M itself is quasi-projective. ◊

In Abbas and Asaad [1], example 3.8, a finitely generated module over a ring (not commutative) was given which was fully d-stable but not quasi-projective.

If the condition (finitely generated) dropped but it is still assumed that M is a direct sum of cyclic modules, and the ring assumed to be (only) commutative without zero divisors, in this case we will also have, full d-stability implies quasi-projectivity. We add the following corollary .

Corollary-2.13.

Let R be an integral domain, M an R-module such that $M = \oplus_{i\in I} M_i$, where $M_i \cong R/I_i$ and I_i is an ideal of R, for each $i \in I$. If M is fully d-stable (multiplication), then it is quasi-projective.

Proof. R is commutative implies it is duo and hence each R/I_i is quasi-projective [4], Lemma 4.1). Therefore M is a direct sum of quasi-projective modules, and it is duo (since it is fully d-stable), this implies M itself is quasi-projective. ◊

A more general result can be added that its proof clear from the above notes.

Proposition-2.14.

Let R be any ring (commutative ring). If a fully d-stable (multiplication) R-module is a direct sum of quasi-projective modules, then it is quasi-projective. ◊

The converse of Proposition 2.14 (in some sense is not true). If M is a quasi-projective and fully d-stable (multiplication) module, then $M \oplus M$ is again quasi-projective but not fully d-stable (multiplication).

On the other hand, for the class of finitely generated torsion modules over a Dedekind domain , The two concepts full d-stability and multiplication coincide and if they hold, the module is quasi-projective. First, recall the following two facts:

A. If M is a finitely generated torsion duo module over a Dedekind domain R, and if M is duo then $M \cong (R/P_1^{n_1}) \oplus \ldots \oplus (R/P_k^{n_k})$ for some positive integers $k, n_1, \ldots, n_k$ and distinct maximal ideals P_i $(1 \le i \le k)$ of R. Ozcan, et al. [3]

B. A torsion module M over a Dedekind domain R, is quasi-projective if and only if each P-primary component M_P is a direct sum copies of the same cyclic R/P^k for some fixed positive integer k depending on P. Ragaswamy and Vanaja [12]

Theorem-2.15.

A finitely generated torsion module over a Dedekind domain is fully d-stable if and only if it is multiplication. In this case the module is quasi-projective.

Proof: It is enough to prove the (only if part). Assume that M is fully d-stable, then it is duo and by (A), $M \cong (R/P_1^{n_1}) \oplus ... \oplus (R/P_k^{n_k})$, where the direct sum on the right satisfies the conditions of Theorem 2.8 and each $R/P_i^{n_i}$ is cyclic and hence multiplication, therefore M is multiplication. By (B) M is quasi-projective. ◊

Note that the Z-module $(Z/2Z) \oplus (Z/2Z)$, for example, is a quasi-projective module which is neither fully d-stable nor multiplication.

To investigate the case when the module is not torsion, first we have the following discussion. Recall that "an R-module M is called divisible, if for each non-zero divisor $s \in R$ and for each $x \in M$, there exists $y \in M$ such that $x = sy$" [13] or equivalently $M = sM$ for each $s \in R$.

Proposition-2.16.

Let R be a ring without zero divisors, but not a division ring. If M is a divisible R-module which is not torsion, then M is not fully d-stable , hence not multiplication.

Proof: Let $0 \neq s \in R$ be non- invertible, then the map $f : M \to M$, defined by $f(x) = sx, \forall x \in M$, is an epimorphism (since $M = sM$). If M is fully d-stable, then it is hopfian [1], hence f is an isomorphism, and has an inverse say g. Let y be a non torsion element of M, then $g(y) = ry$ for some $r \in R$ (since M is duo, see [3]).

Now, $y = f(g(y)) = rsy$, also $y = g(f(y)) = sry$, which implies $rs = sr = 1$(note that we assume y is not torsion), a contradiction with our assumption that s is non-invertible. Hence M cannot be fully d-stable and so it is not multiplication. ◊

From the above proof, we observe that the assumed condition on M can be weakened to be "M (not torsion) is divisible by at least one non-invertible element of R, that is, there exists a non-invertible element $s \in R$, such that $M = sM$" and get the same conclusion.

For the next result we need the following fact about divisible modules.

(C) Over any ring R an injective R-module is divisible. If R is a Dedekind domain, then an R-module is injective if and only if it is divisible. Wisbauer [13]

By (C) any R-module M can be decomposed into $M = D \oplus N$, where D is the largest divisible submodule of M and N has no divisible submodule. A module which has no divisible submodule is called *reduced* (see [14]).

Proposition-2.17.

Let R be a Dedekind domain but not a field. If M is a fully d-stable (multiplication) R-module, then it must be reduced.

Proof: It is enough to prove the case, fully d-stable. If M is not reduced, then (by the above note) it has a non-trivial divisible summand which is not fully d-stable by Proposition 2.16), hence M itself cannot be fully d-stable. ◊

Now, if M is a finitely generated torsion-free R-module(where R is a principal ideal domain), then it is free and hence projective(see [14]) . So, all finitely generated torsion free modules over a principal ideal domain are quasi-projective.

In Abbas and Asaad [2], we mentioned an example of a Z-module which is torsion-free, fully d-stable but not quasi-projective, which was infinitely generated (see [2] , example (2.9)).

Mixed modules (neither torsion nor torsion-free) over a Dedekind domain cannot be quasi-projective [12]. On the other hand, mixed modules, over a Dedekind domain, which has torsion-free direct summand (such modules called M_3, see [15]) are not duo [3], hence not fully d-stable.

If a mixed module exists which is fully d-stable, it must not be M_3, in this case the module is fully d-stable but not quasi-projective.

Proposition-2.18.

A quasi-injective mixed module over a Dedekind domain cannot be duo, hence it is not fully d-stable (not multiplication).

Proof. Let M be a mixed module over a Dedekind domain R, then M has an M_3 submodule (say $Ra \oplus Rb$, where a is torsion-free and b nonzero torsion elements of M) which is not duo, hence M itself is not duo, since " if a duo module is quasi-injective, then any submodule is duo too" [3]. ◊

Corollary-2.19.

If a quasi-injective module over a Dedekind domain is fully d-stable (multiplication), then it is either torsion or torsion free. ◊

In the proof of Proposition 2.18, we use the duo property to get the result. Since a fully stable module is duo too, and it was proved in Abbas [16] that " Every fully stable module over a Dedekind domain is quasi-injective", these facts leads to the following.

Corollary-2.20.

Every fully stable module over a Dedekind domain is either torsion or torsion-free. ◊

In the following, we determine (over a Dedekind domain) the conditions that make quasi-projectivity a necessary condition for full d-stability. First we need the following fact.

(D) A torsion module over a Dedekind domain is quasi-projective if and only if it is quasi-injective but not injective.

Theorem-2.21.

Let M be a quasi-injective torsion module over a Dedekind domain. If M is fully d-stable (multiplication), then it is quasi-projective.

Proof: By Proposition 2.17 and (C) M is not injective and by (D) it is quasi-projective. ◊

The torsion-free case (over Dedekind domain) was just discussed, but under the finitely generated condition, (see the remark after Proposition 2.17). The condition of finitely generated cannot be dropped, since we had an example of a torsion-free Z-module, fully d-stable but not quasi-projective which was not finitely generated, (see example 2.9, [2]) .

Other result on wider class , namely , non torsion modules over integral domains will be discussed in the next. In Abbas and Asaad [1], it was proved " If M is a duo torsion-free module over an integral domain R, and $f \in End_R(M)$, then there exists $r \in R$ such that $f(m) = rm$ for each $m \in M$ " (see Proposition 2.13, [1]). This statement will be generalized to include all non torsion modules.

Proposition-2.22.

If M is a non torsion duo module over an integral domain R, and $f \in End_R(M)$, then there exists $r \in R$ such that $f(m) = rm$ for each $m \in M$.

Proof: Let $f \in End_R(M)$, and $T(M)$ be the torsion submodule of M. First we will prove that there exists $r \in R$ such that $f(m) = rm$ for each $m \notin T(M)$ (the same argument used in the proof of (Proposition 2.13, [1]) will be repeated) . Let x, y be two non torsion elements of M, then there exist $r, s \in R$ such that $f(x) = rx, f(y) = sy$ (see Lemma 1.1,[14]). It is claimed that $r = s$. There are two cases:

(i) $Rx \cap Ry \neq 0$. Let $0 \neq z \in Rx \cap Ry$, assume $f(z) = tz, t \in R$, then $z = ux = vy$ for some $u, v \in R$, hence $tz = f(ux) = urx$ which implies $tux = urx$ and so $t = r$ (since x is not torsion). Similarly $t = s$ (since y is not torsion too). Therefore $r = s$.

(ii) $Rx \cap Ry = 0$, assume $f(x + y) = t(x + y)$, $t \in R$, but $f(x + y) = rx + sy$, so we have $(t - r)x = (s - t)y$ which implies $t - r = s - t = 0$, that is $r = s$.

Therefore $f(m) = rm$ for each $m \notin T(M)$, where $r \in R$ is fixed.

Now fix $x \notin T(M)$ and let $z \in T(M)$, then $x+z \notin T(M)$, so by the above part of the proof, $f(x+z)=r(x+z)$, then $f(x)+f(z)=rx+rz$, hence $f(z)=rz$. Therefore $f(m)=rm$ for each $m \in M$. ◊

Now, we can generalize Corollary 2.15 in Abbas and Asaad [1] that says " If M is a fully d-stable module over an integral domain R, and N is a submodule of M such that M/N is torsion-free, then for every homomorphism $\alpha : M \to M/N$ there exists $r \in R$ such that $\alpha(m)=rm+N$ for each $m \in M$ " in the following, and omit the proof that, since it uses the same argument in the proof of the original statement.

Corollary-2.23.

If M is a fully d-stable module over an integral domain R, and N is a submodule of M such that M/N is not torsion, then for every homomorphism $\alpha : M \to M/N$ there exists $r \in R$ such that $\alpha(m)=rm+N$ for each $m \in M$. ◊

The following corollary gives a partially answer to our main question in certain conditions.

Corollary-2.24.

If M is a fully d-stable module over an integral domain R, and N is a submodule of M such that M/N is not torsion, then every homomorphism $\alpha : M \to M/N$ can be lifted to an endomorphism of M, that is , there exists $f \in End_R(M)$ such that $f \circ \pi = \alpha$, where π is the natural epimorphism of M onto M/N.

Proof: By Corollary 2.23 $\alpha(m)=rm+N$, for some $r \in R$, hence f, defined by $f(m)=rm$ is the desired endomorphism. ◊

Now we ready to state and prove the following theorem.

Theorem-2.25.

Let M be a torsion-free module over an integral domain R, and M has no proper essential submodule. If M is fully d-stable, then it is quasi-projective.

Proof: First we will show, that the hypothesis of the theorem leads to the following result: if N is any proper submodule, then M/N is not torsion, since (by the hypothesis) there exists $0 \neq m \in M$ such that $Rm \cap N = 0$ which means that $m + M$ is not torsion, and hence M/N is not torsion. Therefore, by Corollary 2.24 M is quasi-projective. ◊

Note that, the hypothesis of Theorem 2.25, can be expand to include the modules over integral domains with the property that M/N is not torsion for any proper submodule N.

3. CONCLUSION

In the following we summarize the cases (investigated in this work) that full d-stability implies quasi-projectivity.

1. A finitely generated torsion module over a Dedekind domain.
2. A direct sum of the form $M = \oplus_{i \in I} M_i$, where $M_i \cong R/I_i$ and I_i is an ideal of R, for each $i \in I$, and R is an integral domain.
3. A direct sum of quasi-projective modules.
4. A quasi-injective torsion module over a Dedekind domain.
5. A finitely generated torsion free module over a principal ideal domain . These type of modules are quasi-projective without the need of full d-stability condition.
6. A torsion-free module over an integral domain, which has no proper essential submodule.

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