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WEAK SEPARATION AXIOMS VIA $\Omega-$OPEN SET AND $\Omega-$CLOSURE OPERATOR

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ABSTRACT

In this paper we introduce a new type of weak separation axioms with some related theorems and show that they are equivalent with these in [1].

Keywords: Weak separation axioms, Weak $\omega-$open sets, Weak regular spaces, $\omega-ker(A)$, Sober $\omega-R_0$, $\omega-R_0$ space, $\omega-R_1$ space.

1. INTRODUCTION

In this article let us prepare the background of the subject. Throughout this paper , (X,T) stands for topological space. Let A be a subset of X. A point x in X is called *condensation* point of A if for each U in T with x in U, the set $U \cap A$ is uncountable [2]. In 1982 the $\omega-$closed set was first introduced by Hdeib [2], and he defined it as: A is $\omega-$*closed* if it contains all its condensation points and the $\omega-$*open* set is the complement of the $\omega-$closed set. It is not hard to prove: any open set is $\omega-$open. Also we would like to say that the collection of all $\omega-$open subsets of X forms topology on X. The closure of A will be denoted by $cl(A)$, while the intersection of all $\omega-$closed sets in X which containing A is called the $\omega-$*closure* of A, and will denote by $cl_\omega(A)$. Note that $cl_\omega(A) \subset cl(A)$.

In 2005 Caldas, et al. [3] introduced some weak separation axioms by utilizing the notions of $\delta-pre-$open sets and $\delta-pre-$closure. In this paper we use Caldas, et al. [3] definitions to introduce new spaces by using the $\omega-$open sets defined by Hdeib [2], we ecall it $\omega-R_i-$Spaces $i=0,1,2$, and we show that $\omega-R_0$, $\omega^\star-T_1$ space and $\omega-$symmetric space are equivalent.

For our main results we need the following definitions and results:

Definition-1.1.

Noiri, et al. [4] A space (X,T) is called a *door space* if every subset of X is either open or closed.

Definition-1.2.

Hadi [1] The topological space X is called $\omega^{\star}-T_1$ *space* if and only if, for each $x \neq y \in X$, there exist ω −open sets U and V, such that $x \in U, y \notin U$, and $y \in V, x \notin V$.

Lemma-1.3.

Hadi [1] The topological X is $\omega^{\star}-T_1$ if and only if for each $x \in X, \{x\}$ is ω −closed set in X.

Definition-1.4.

Hadi [1] The topological space X is called $\omega^{\star}-T_2$ *space* if and only if, for each $x \neq y \in X$, there exist two disjoint ω −open sets U and V with $x \in U$ and $y \in V$.

For our main result we need the following property of ω −closure of a set:

Proposition-1.5.

Let $\{A_\lambda, \lambda \in \Lambda\}$ be a family of subsets of the topological space (X, T), then

1. $cl_\omega(\cap_{\lambda\in\Lambda} A_\lambda) \subseteq \cap_{\lambda\in\Lambda} cl_\omega(A_\lambda)$.
2. $\cup_{\lambda\in\Lambda} cl_\omega(A_\lambda) \subseteq cl_\omega(\cup_{\lambda\in\Lambda} A_\lambda)$.

Proof:

1. It is clear that $\cap_{\lambda\in\Lambda} A_\lambda \subseteq A_\lambda$ for each $\lambda \in \Lambda$. Then by (4) of Theorem 1.5.3 in Hadi [1], we have $cl_\omega(\cap_{\lambda\in\Lambda} A_\lambda) \subseteq cl_\omega(A_\lambda)$ for each $\lambda \in \Lambda$. Therefore $cl_\omega(\cap_{\lambda\in\Lambda} A_\lambda) \subset \cap_{\lambda\in\Lambda} cl_\omega(A_\lambda)$.

Note that the opposite direction is not true . For example consider the usual topology T for $\mathbb{R}$, If $A_i = \left(0, \frac{1}{i}\right), i = 1,2, \ldots,$ and $\cap_{i\in\mathbb{N}} cl_\omega(A_i) = \{0\}$. But $cl_\omega(\cap_{i\in\mathbb{N}} A_i) = cl_\omega(\emptyset) = \emptyset$. Therefore $\cap_{\lambda\in\Lambda} cl_\omega(A_\lambda) \nsubseteq cl_\omega(\cap_{\lambda\in\Lambda} A_\lambda)$.

2. Since $A_\lambda \subseteq \cup_{\lambda\in\Lambda} A_\lambda$, for each $\lambda \in \Lambda$. Then by (4) of Theorem 1.5.3 in Hadi [1], we get $cl_\omega(A_\lambda) \subseteq cl_\omega(\cup_{\lambda\in\Lambda} A_\lambda)$, for each $\lambda \in \Lambda$. Hence $\cup_{\lambda\in\Lambda} cl_\omega(A_\lambda) \subseteq cl_\omega(\cup_{\lambda\in\Lambda} A_\lambda)$.

Note that the opposite direction is not true . For example consider the usual topology T for $\mathbb{R}$, If $A_i = \left\{\frac{1}{i}\right\}, i = 1,2, \ldots,$, $cl_\omega(A_i) = \left\{\frac{1}{i}\right\}$, and $\cup_{i\in\mathbb{N}} cl_\omega(A_i) = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$. But $cl_\omega(\cup_{i\in\mathbb{N}} A_i) = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, 0\right\}$. Thus $cl_\omega(\cup_{\lambda\in\Lambda} A_\lambda) \nsubseteq \cup_{\lambda\in\Lambda} cl_\omega(A_\lambda)$ □

2. $\Omega - R_I$ −SPACES , FOR $i = 0, 1$

In this section we introduce some types of weak separation axioms by utilizing the ω −open sets defined in Hdeib [2].

Definition-2.1.

Let $A \subset (X, T)$, then the ω −*kernal* of A denoted by $\omega - ker(A)$ is the set

$\omega - ker(A) = \cap\{$O,where O is an ω- open set in (X, T) containing $A\}$.

Proposition-2.2.

Let $A \subset (X, T)$, and $x \in X$. Then

$\omega - ker(A) = \{x \in X : cl_\omega(\{x\}) \cap A \neq \emptyset\}$.

Proof:

Let A be a subset of X, and $x \in \omega - ker(A)$, such that $cl_\omega(\{x\}) \cap A = \emptyset$. Then $x \notin X \backslash cl_\omega(\{x\})$, which is an ω −open set containing A. This contradicts $x \in \omega - ker(A)$. So $cl_\omega(\{x\}) \cap A \neq \emptyset$. Then Let $x \in X$, be a point satisfied $cl_\omega(\{x\}) \cap A \neq \emptyset$. Assume $x \notin \omega - ker(A)$, then there exists an ω −open set G containing A but not x. Let $y \in cl_\omega(\{x\}) \cap A$. Hence G is an ω −open set containing y but not x. This contradicts $cl_\omega(\{x\}) \cap A \neq \emptyset$. So $x \in \omega - ker(A)$ □

Definition-2.3.

A topological space (X, T) is said to be *sober* $\omega - R_0$ if $\cap_{x\in X} cl_\omega(\{x\}) = \emptyset$.

Theorem-2.4.

A topological space (X, T) is sober $\omega - R_0$ if and only if $\omega - ker(\{x\}) \neq X$ for each $x \in X$.

Proof:

Suppose that (X, T) is sober $\omega - R_0$. Assume there is a point $y \in X$, with $\omega - ker(\{y\}) = X$. Let $x \in X$, then $x \in V$ for any $\omega -$ open set V containing y , so $y \in cl_\omega(\{x\})$ for each $x \in X$. This implies $y \in \cap_{x\in X} cl_\omega(\{x\})$, which is a contradiction with $\cap_{x\in X} cl_\omega(\{x\}) = \emptyset$.

Now suppose ω −kernal$(\{x\}) \neq X$ for every $x \in X$. Assume X is not sober $\omega - R_0$, it mean there is y in X such that $y \in \cap_{x\in X} cl_\omega(\{x\})$, then every $\omega -$ open set containing y must contain every point of X. This implies that X is the unique ω- open set containing y. Therefore ω −kernal$(\{y\}) = X$, which is a contradiction with our hypothesis. Hence (X, T) is sober $\omega - R_0$ □

Definition-2.5.

A map $f: X \to Y$ is called ω −*closed*, if the image of every ω −closed subset of X is ω −closed in Y.

Proposition-2.6.

If X is a space, f is a map defined on X and $A \subseteq X$, then
$cl_\omega(f(A)) \subseteq f(cl_\omega(A)$.

Proof:

We have $A \subseteq cl_\omega(A)$, then $f(A) \subseteq f(cl_\omega(A))$ This implies $cl_\omega(f(A)) \subseteq cl_\omega(f(cl_\omega(A))) = f(cl_\omega(A))$. Hence $cl_\omega(f(A)) \subseteq f(cl_\omega(A)$ □

Theorem-2.7.

If $f: X \to Y$ is one to one ω −closed map and X is sober $\omega - R_0$, then Y is sober $\omega - R_0$.

Proof:

From Proposition 1.5, we have

$$\begin{aligned} \cap_{y\in Y} cl_\omega(\{y\}) \subset \cap_{x\in X} cl_\omega(\{f(x)\}) \subset \cap_{x\in X} f(cl_\omega(\{x\})) \\ = f(\cap_{x\in X} cl_\omega(\{x\})) \\ = f(\emptyset) = \emptyset. \end{aligned}$$

Thus Y is sober $\omega - R_0$

Definition-2.8.

A topological space (X,T) is called $\omega - R_0$ if every ω −open set contains the ω −closure of each of its singletons.

Theorem-2.9.

The topological door space is $\omega - R_0$ if and only if it is $\omega^\star - T_1$.

Proof:

Let x, y are distinct points in X. Since (X,T) is door space so that for each x in , $\{x\}$ is open or closed.

i. 1. When $\{x\}$ is open, hence ω −open set in X. Let $V = \{x\}$, then $x \in V$, and $y \notin V$. Therefore since (X,T) is $\omega - R_0$ space, so that $cl_\omega(\{x\}) \subset V$. Then $x \notin X\backslash V$, while $y \in X\backslash V$, where $X\backslash V$ is an ω −open subset of X.

2. Whenever $\{x\}$ is closed**,** hence it is ω −closed, $y \in X\backslash\{x\}$, and $X\backslash\{x\}$ is ω −open set in X**.** Then since (X,T) is $\omega - R_0$ space, so that $cl_\omega(\{y\}) \subset X\backslash\{x\}$. Let $V = X\backslash\ cl_\omega(\{y\})$, then $x \in V$, but $y \notin V$, and V is an ω −open set in X. Thus we obtain (X,T) is $\omega^\star - T_1$.

ii. For the other direction assume (X,T) is $\omega^\star - T_1$, and let V be an ω −open set of X, and $x \in V$. For each $y \in X\backslash V$, there is an ω −open set V_y such that $x \notin V_y$, but $y \in V_y$. So $cl_\omega(\{x\}) \cap V_y = \emptyset$, which is true for each $y \in X\backslash V$. Therefore $cl_\omega(\{x\}) \cap (\bigcup_{y \in X\backslash V} V_y) = \emptyset$**.** Then since $y \in V_y$, $X\backslash V \subset \bigcup_{y\in X\backslash V} V_y$, and $cl_\omega(\{x\}) \subset V$. Hence (X,T) is $\omega - R_0$ □

Definition-2.10.

A topological space (X,T) is ω −*symmetric* if for x and y in the space X, $x \in cl_\omega(\{y\})$ implies $y \in cl_\omega(\{x\})$.

Proposition-2.11.

Let X be a door ω −symetric topological space . Then for each $x \in X$, the set $\{x\}$ is ω −closed.

Proof:

Let $x \neq y \in X$, since X is a door space so $\{y\}$ is open or closed set in X. When $\{y\}$ is open, so it is ω −open, let $V_y = \{y\}$. Whenever $\{y\}$ is ω −closed , $x \notin \{y\} = cl_\omega(\{y\})$. Since X is ω −symetric we get $y \notin cl_\omega(\{x\})$. Put $V_y = X\backslash cl_\omega(\{x\})$, then $x \notin V_y$ and $y \in V_y$, and V_y is ω −open set in X. Hence we get for each $y \in X\backslash\{x\}$ there is an ω −open set V_y such that $x \notin V_y$ and $y \in V_y$. Therefore $X\backslash\{x\} = \bigcup_{y\in X\backslash\{x\}} V_y$ is ω −open, and $\{x\}$ is ω −closed □

Proposition-2.12.

Let (X,T) be $\omega^\star - T_1$ topological space, then it is ω −symetric space.

Proof:

Let $x \neq y \in X$. Assume $y \notin cl_\omega(\{x\})$, then since X is $\omega - T_1$ there is an open set U containing x but not y, so $x \notin cl_\omega(\{y\})$. This completes the proof □

Theorem-2.13.

The topological door space is $\omega -$ symmetric if and only if it is $\omega^\star - T_1$.

Proof:

Let (X, T) be a door $\omega -$ symmetric space. Then using Proposition 2.11 for each $\in X$, $\{x\}$ is $\omega -$closed set in X. Then Lemma 1.3, we get that (X, T) is $\omega^\star - T_1$. On the other hand, assume (X, T) is $\omega^\star - T_1$, then directly by Proposition 2.12. (X, T) is $\omega -$ symmetric space □

Corollary-2.14.

Let (X, T) be a topological door space, then the following are equivalent:

1. (X , T) is $\omega - R_0$ space.

2. (X , T) is $\omega^\star - T_1$ space.

3. (X , T) is $\omega -$ symmetric space.

Proof:

The proof follows immediately from Theorem 2.9 and Theorem 2.13

□

Corollary-2.15.

If (X, T) is a topological door space, then it is $\omega - R_0$ space if and only if for each $x \in X$, the set $\{x\}$ is $\omega -$closed set.

Proof:

We can prove this corollary by using Corollary 2.14 and Lemma 1.3

Theorem-2.16.

Let (X, T) be a topological space contains at least two points. If X is $\omega - R_0$ space, then it is sober $\omega - R_0$ space.

Proof:

Let x and y are two distinct points in X. Since (X, T) is $\omega - R_0$ space so by Theorem 2.8 it is $\omega^\star - T_1$. Then Lemma 1.3 implies $cl_\omega(\{x\}) = \{x\}$ and $cl_\omega(\{y\}) = \{y\}$. Therefore $\cap_{p\in X} \ cl_\omega(\{p\}) \subset cl_\omega(\{x\}) \cap cl_\omega(\{y\}) = \{x\}\cap\{y\} = \emptyset$. Hence (X, T) is sober $\omega - R_0$ space.

Definition-2.17.

A topological door space (X, T) is said to be $\omega - R_1$ ***space*** if for x and y in , with $cl_\omega(\{x\}) \neq cl_\omega(\{y\})$, there are disjoint $\omega -$open set U and V such that $cl_\omega(\{x\}) \subset U$, and $cl_\omega(\{y\}) \subset V$.

Theorem-2.18.

The topological door space is $\omega - R_1$ if and only if it is $\omega^\star - T_2$ space.

Proof:

Let x and y be two distinct points in X. Since X is door space so
for each x in X, The set $\{x\}$ is open or closed.

i. If $\{x\}$ is open. Since $\{x\}\cap\{y\} = \emptyset$, then $\{x\}\cap cl_\omega\{y\} = \emptyset$. Thus $cl_\omega(\{x\}) \neq cl_\omega(\{y\})$.

ii. Whenever $\{x\}$ is closed, so it is $\omega-$closed and $cl_{\omega}\{x\}\cap\{y\}=\{x\}\cap\{y\}=\emptyset$. Therefore $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$. We have (X, T) is $\omega-R_1$ space, so that there are disjoint $\omega-$open sets U and V such that $x\in cl_{\omega}(\{x\})\subset U$, and $y\in cl_{\omega}(\{y\})\subset V$, so X is $\omega^{\star}-T_2$ space.

For the opposite side let x and y be any points in X, with $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$. Since every $\omega^{\star}-T_2$ space is $\omega^{\star}-T_1$ space so by (3) of Theorem 2.2.15 $cl_{\omega}(\{x\})=\{x\}$ and $cl_{\omega}(\{y\})=\{y\}$, this implies $x\neq y$. Since X is $\omega^{\star}-T_2$ there are two disjoint $\omega-$open sets U and V such that $cl_{\omega}(\{x\})=\{x\}\subset U$, and $cl_{\omega}(\{y\})=\{y\}\subset V$. This proves X is $\omega-R_1$ space □

Corollary-2.19.

Let (X,T) be a topological door space. Then if X is $\omega-R_1$ space then it is $\omega-R_0$ space.

Proof:

Let X be an $\omega-R_1$ door space. Then by Theorem 2.17 X is $\omega^{\star}-T_2$ space. Then since every $\omega^{\star}-T_2$ space is $\omega^{\star}-T_1$, so that by Theorem 2.9, X is $\omega-R_0$ space.

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