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THREE FASCINATING PAIRS

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ABSTRACT

This paper concerns with obtaining two non-zero different integers N_1 and N_2 such that

(i) $N_1 - N_2 = \alpha^2$, $N_1 N_2 = 5\beta^2$

(ii) $N_1 - N_2 = (k^2+1)\alpha^2$, $N_1 N_2 = k^2\beta^2$

(iii) $N_1 - N_2 = (6n - 6n^2 + 1)$, $N_1 N_2 = k^2$

respectively in sections A,B and C.

Contribution/ Originality: This paper concerns with the problem of investigating integer solutions to a special system of two equations with two unknowns.

1. INTRODUCTION

Number theory along with Geometry [1] is one of the two oldest branches of Mathematics. Number theory, as a fundamental body of knowledge, has played a significant role in the development of Mathematics. The study of Number theory is elegant, beautiful and delightful. In fact, Number theory is that branch of mathematics which deals with the properties of integers, more specifically, the properties of positive integers. These numbers, together with the negative integers and zero form the set of integers. Properties of these numbers have been studied from the earliest times [2-5]. It has fascinated and inspired both amateurs and mathematicians alike. Diophantine problems have fewer equations than unknown variables and involve finding solutions in integers [6-11].

In this communication, we attempt for obtaining two non-zero distinct integers N_1 and N_2 such that

(i) $N_1 - N_2 = \alpha^2$, $N_1 N_2 = 5\beta^2$

(ii) $N_1 - N_2 = (k^2+1)\alpha^2$, $N_1 N_2 = k^2\beta^2$

(iii) $N_1 - N_2 = (6n - 6n^2 + 1)$, $N_1 N_2 = k^2$

2. METHOD OF ANALYSIS

SECTION A:

Let N_1 and N_2 be any two non-zero distinct integers such that

$$N_1 - N_2 = \alpha^2 \tag{1}$$

$$N_1 N_2 = 5\beta^2 \tag{2}$$

Eliminating N_2 between (1) and (2) , we get

$$N_1^2 - \alpha^2 N_1 - 5\beta^2 = 0.$$

Treating this as a quadratic in N_1 and solving for N_1 , we have

$$N_1 = \frac{1}{2}\left[\alpha^2 + \sqrt{\alpha^4 + 20\beta^2}\right] \quad \text{(Taking positive sign)} \tag{3}$$

Let $U^2 = 20\beta^2 + \alpha^4$ (4)

Choice-1.

Equation (4) is written as the system of two equations as shown in the table 1 below:

system	1	2	3	4
$U+\alpha^2$	$2\beta^2$	5β	10β	$10\beta^2$
$U-\alpha^2$	10	4β	2β	2

Solving each of the above system for α, β, U and employing (3) and (1), the corresponding non-zero integer solutions satisfying (1) and (2) are obtained as shown below:

Case-1.

The system of equations

$$U + \alpha^2 = 2\beta^2$$

$$U - \alpha^2 = 10$$

give $\alpha = 2, \beta = 3, U = 14$

Thus, in view of (2) and (3) we have

$$N_1 = 9, -5$$

$$N_2 = 5, -9$$

which represent the required values of N_1 and N_2 satisfying (1) and (2).

Case-2.

Solving the pair

$$U+\alpha^2=5\beta$$
$$U-\alpha^2=4\beta$$

we obtain $\alpha=s,\beta=2s^2$

Thus, in view of (2) and (3) we have

$$N_1=5s^2,-4s^2$$
$$N_2=4s^2,-5s^2$$

which represent the required values of N_1 and N_2 satisfying (1) and (2).

Case-3.

From the double equations

$$U+\alpha^2=10\beta$$
$$U-\alpha^2=2\beta$$

we obtain $\alpha=2s,\beta=s^2$

Thus , in view of (2) and (3) we have

$$N_1=5s^2,-s^2$$
$$N_2=s^2,-5s^2$$

which represent the required values of N_1 and N_2 satisfying (1) and (2).

Case-4.

$$U+\alpha^2=10\beta^2$$
$$U-\alpha^2=2$$

On solving the above two equations, we obtain $\alpha^2=5\beta^2-1$ (5)

with the least positive integer solutions $\beta_0=1,\alpha_0=2$

To obtain the other solutions of equation (5) ,Consider the Pellian equation

$$\alpha^2=5\beta^2+1$$

whose general solution,

$$\tilde{\alpha}_n=\frac{1}{2}f_n,\tilde{\beta}_n=\frac{1}{2\sqrt{5}}g_n$$

in which $f_n=\left(9+4\sqrt{5}\right)^{n+1}+\left(9-4\sqrt{5}\right)^{n+1}$

$g_n=\left(9+4\sqrt{5}\right)^{n+1}-\left(9-4\sqrt{5}\right)^{n+1}$, where n= -1, 0, 1, 2……

Applying Brahmagupta lemma between the solutions of (α_0, β_0) and $(\tilde{\alpha}_n, \tilde{\beta}_n)$ the general solutions of equation (5) are found to be

$$\alpha_{n+1} = f_n + \frac{\sqrt{5}g_n}{2}$$

$$\beta_{n+1} = \frac{1}{2} f_n + \frac{g_n}{\sqrt{5}}$$

Thus, in view of (2) and (3) we have

$$N_1 = 5\left(\frac{1}{2} f_n + \frac{g_n}{\sqrt{5}}\right)^2, -1$$

$$N_2 = 1, -5\left(\frac{1}{2} f_n + \frac{g_n}{\sqrt{5}}\right)^2$$

which represent the required values of N_1 and N_2 satisfying (1) and (2).

Table-2. Numerical examples

n	$\frac{1}{2} f_n$	$\frac{g_n}{2\sqrt{5}}$	N_1	N_2	$N_1 - N_2$	$N_1 N_2$
0	9	4	1445	1	38^2	5×17^2
1	161	72	465125	1	682^2	5×305^2
2	2889	1292	149768645	1	12238^2	5×5473^2

Source: Manual

Choice-2.

(4) is satisfied by $\beta = 2rs, \alpha^2 = 20r^2 - s^2, U = 20r^2 + s^2$

Consider the equation $\alpha^2 = 20r^2 - s^2$ (6)

Introducing the linear transformations

$\alpha = u + v, s = u - v (u \neq v \neq 0)$ (7)

It leads to $u^2 + v^2 = 10r^2$ (8)

The above equation is solved through different methods and employing (7), different sets of distinct integer solutions to (1) and (2) are obtained which are illustrated below:

Case-1.

Write 10 as $10 = (1+3i)(1-3i)$ (9)

Assume $r = a^2 + b^2$ (10)

where a and b are non zero distinct integers

Using (9) & (10) in (8) and employing the method of factorization, define

$$u + iv = (1+3i)(a+ib)^2$$

from which, on equating the real and imaginary parts

$$u = a^2 - b^2 - 6ab$$

$$v = 3a^2 - 3b^2 + 2ab$$

Substituting the above values of u and v in (7), the values of α and s are given by

$$\alpha = 4a^2 - 4b^2 - 4ab$$
$$s = -2a^2 + 2b^2 - 8ab \qquad (11)$$

and $U = 20(a^2+b^2)^2 + 4(b^2 - a^2 - 4ab)^2$ (12)

Thus, in view of (2), (3) and (4) we have

$$N_1 = 20(a^2+b^2)^2, 32ab(b^2-a^2) - 4b^4 - 4a^4 - 56a^2b^2$$
$$N_2 = (4b^4 + 4a^4 + 56a^2b^2 - 32ab(b^2-a^2), -20(a^2+b^2)^2$$

which represent the required values of N_1 and N_2 satisfying (1) and (2).

Case-2.

In addition to (9), one may write 10 as $10 = (3+i)(3-i)$

For this choice, the corresponding integer solutions to (1) and (2) are given by

$$N_1 = 20(a^4+b^4) + 40a^2b^2, 32ab(a^2-b^2) - 4b^4 - 4a^4 - 56a^2b^2$$
$$N_2 = 4(a^4+b^4) + 56a^2b^2 - 32ab(a^2-b^2), -20a^4 - 20b^4 - 40a^2b^2$$

Section-B:

Consider the system of double equations given by

$$N_1 - N_2 = (k^2+1)\alpha^2 \qquad (13)$$

$$N_1 N_2 = k^2 \beta^2 \qquad (14)$$

where k, α, β are non-zero integers.

At the outset, note that the system of equations of (13) and (14) is satisfied by

$$N_1 = (k^6 - k^2)s^2, N_2 = (k^4 - 1)s^2$$

However, we have other pairs of (N_1, N_2) satisfying the system of equations (13) and (14) and they are obtained as shown below:

Eliminating N_2 between (13) and (14), we get

$$N_1^2-(k^2+1)\alpha^2 N_1-k^2\beta^2=0$$

Treating this as a quadratic in N_1 and solving for N_1 , we have

$$N_1=\frac{1}{2}\left[(k^2+1)\alpha^2\pm\sqrt{(k^2+1)^2\alpha^4+4k^2\beta^2}\right]$$

Taking $\beta=(k^2+1)B$, we have

$$N_1=\frac{(k^2+1)}{2}\left[\alpha^2\pm\sqrt{\alpha^4+4k^2B^2}\right] \quad (15)$$

Case:1 Let $B>k$

The square root on the R.H.S of (15) is eliminated provided

$$\alpha^2=B^2-k^2 \quad (16)$$

which is satisfied by $\alpha=2rs, k=r^2-s^2, B=r^2+s^2, r>s>0$

Substituting the above values of α, k, B in (15) and using (13), the corresponding two pairs of (N_1,N_2) are given by

$$(N_1,N_2)=\begin{Bmatrix}\left((r^2+s^2)^2((r^2-s^2)^2+1),((r^2-s^2)^2+1)(r^2-s^2)^2\right),\\ \left(-((r^2-s^2)^2+1)(r^2-s^2)^2,-(r^2+s^2)^2((r^2-s^2)^2+1)\right)\end{Bmatrix}$$

Table-3. Numerical examples

r	**s**	α	**k**	**B**	β	N_1	N_2	N_1-N_2	N_1N_2
2	1	4	3	5	50	250	90	$(3^2+1)4^2$	$3^2\times 50^2$
3	2	12	5	13	338	4394	650	$(5^2+1)12^2$	$5^2\times 338^2$
3	1	6	8	10	650	6500	4160	$(8^2+1)6^2$	$8^2\times 650^2$
4	2	16	12	20	2900	58000	20880	$(12^2+1)16^2$	$12^2\times 2900^2$

Source: Manual

Note that, the solutions to (16) are also written as

$$\alpha=r^2-s^2, k=2rs, B=r^2+s^2, r>s>0$$

The corresponding two pairs of (N_1,N_2) are as shown below:

$$(N_1,N_2)=\begin{Bmatrix}\left((4r^2s^2+1)(r^2-s^2)^2,(4r^2s^2+1)4r^2s^2\right),\\ \left(-4r^2s^2(4r^2s^2+1),-(4r^2s^2+1)(r^2-s^2)^2\right)\end{Bmatrix}$$

Case:2 Let $k>B$

The square root on the R.H.S of (15) is eliminated provided

$$\alpha^2 = k^2 - B^2 \tag{17}$$

which is satisfied by $\alpha = 2rs, B = r^2 - s^2, r = r^2 + s^2, r > s > 0$

Substituting the above values of α, k, B in (15) and using (13), the corresponding two pairs of (N_1, N_2) are given by

$$(N_1, N_2) = \left\{ \begin{matrix} \left((r^2+s^2)^2((r^2-s^2)^2+1), ((r^2-s^2)^2+1)(r^2-s^2)^2\right), \\ \left(-((r^2-s^2)^2+1)(r^2-s^2)^2, -(r^2+s^2)^2((r^2-s^2)^2+1)\right) \end{matrix} \right\}$$

Table-4. Numerical examples

r	*s*	α	*k*	B	β	N_1	N_2	$N_1 - N_2$	$N_1 N_2$
2	1	3	4	5	85	425	272	$(4^2+1)3^2$	$17^2 \times 4^2 \times 5^2$
3	2	5	12	13	1885	24505	20880	$(12^2+1)5^2$	$12^2 \times 13^2 \times 145^2$
3	1	8	6	10	370	3700	1332	$(6^2+1)8^2$	$6^2 \times 10^2 \times 37^2$
4	2	12	16	20	5140	102800	65792	$(16^2+1)12^2$	$12^2 \times 16^2 \times 257^2$

Source: Manual

Note that, the solutions to (16) are also written as

$$\alpha = r^2 - s^2, B = 2rs, k = r^2 + s^2, r > s > 0$$

The corresponding two pairs of (N_1, N_2) are as shown below:

$$(N_1, N_2) = \left\{ \begin{matrix} \left(((r^2+s^2)^2+1)(r^2+s^2)^2, -((r^2+s^2)^2+1)4r^2s^2\right), \\ \left(((r^2+s^2)^2+1)4r^2s^2, -((r^2+s^2)^2+1)(r^2+s^2)^2\right) \end{matrix} \right\}$$

SECTION C:

Consider the system of double equations given by

$$N_1 - N_2 = (6n^2 - 6n + 1) \tag{18}$$

$$N_1 N_2 = k^2 \tag{19}$$

where n, k are non-zero integers.

Eliminating N_2 between (18) and (19) , we get

$$N_1^2 - (6n^2 - 6n + 1)N_1 - k^2 = 0.$$

Treating this as a quadratic in N_1 and solving for N_1 , we have

$$N_1 = \frac{1}{2}\left[(6n^2 - 6n + 1) \pm \sqrt{(6n^2 - 6n + 1)^2 + 4k^2}\right] \qquad (20)$$

Let $\alpha^2 = (6n^2 - 6n + 1)^2 + 4k^2$

which is satisfied by $k = 2rs, 6n^2 - 6n + 1 = r^2 - s^2, \alpha = r^2 + s^2$ (21)

Substitute $r = 3n^2 - 3n + 1, s = 3n^2 - 3n$ in (21) and performing few calculations, in view of (20) and (18), the corresponding two pairs of (N_1, N_2) are found to be

$$(N_1, N_2) = \left((3n^2 - 3n + 1)^2, (3n^2 - 3n)^2\right)$$

Table-5. Numerical examples

s.no	n	N_1	N_2	$N_1 - N_2$	N_1N_2
1	2	49	36	13	42^2
2	3	361	324	37	342^2
3	4	1369	1296	73	1332^2
4	5	3721	3600	121	3660^2

Source: Manual

Remark:

$N_1 + N_2$ is written as sum of two squares in 2 ways . Hence $N_1 + N_2$ is a R_2 number.

3. CONCLUSION

In this paper two different non-zero integers N_1 and N_2 are obtained such that

(i) $N_1 - N_2 = \alpha^2$, $N_1N_2 = 5\beta^2$

(ii) $N_1 - N_2 = (k^2 + 1)\alpha^2$, $N_1N_2 = k^2\beta^2$

(iii) $N_1 - N_2 = (6n - 6n^2 + 1)$, $N_1N_2 = k^2$

As Diophantine problems are infinitely many, a search may be made for finding other forms of Diophantine problems.

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