Journal of Asian Scientific Research
ISSN(e): 2223-1331
ISSN(p): 2226-5724
DOI: 10.18488/journal.2.2018.82.61.72
Vol. 8, No. 2, 61-72

URL: www.aessweb.com

SOME PROPERTIES OF SEVERAL PROOF SYSTEMS FOR INTUITIONISTIC, JOHANSSON'S AND MONOTONE PROPOSITIONAL LOGICS

Chubaryan Anahit[1+]
Karabakhtsyan Arman[2]
Petrosyan Garik[3]

[1]*Doctor of Physical and Mathematical Sciences, Full Professor of Yerevan State University and Russian-Armenian University, Armenia*
[2]*Master Student of Department of Applied Mathematics and Informatics of Russian-Armenian University, Armenia*
[3]*Master Student of Department of Informatics and Applied Mathematics of Yerevan State University, Armenia*

(+ Corresponding author)

Article History
Received: 19 December 2017
Revised: 19 January 2018
Accepted: 23 January 2018
Published: 30 January 2018

Keywords
Strongly equal tautology
Minimal tautology
Sequent proof systems
Frege systems
Proof complexity measures
Monotonous system.

ABSTRACT

In this paper we investigate two properties of some propositional systems of Intuitionistic, Johansson's and Monotone logics: 1) the relations between the proofs complexities of strongly equal tautologies (valid sequents) and 2) the relations between the proofs complexities of minimal tautologies (valid sequents) and of results of substitutions in them. We show that 1) strongly equal tautologies (valid sequents) can have essential different proof complexities in the same system and 2) the result of substitution can be proved easier, than corresponding minimal tautology (valid sequents), therefore the systems, which are considered in this paper, are no monotonous neither by lines nor by size.

1. INTRODUCTION

The traditional assumption that all tautologies as Boolean functions are equal to each other is not fine-grained enough to support a sharp distinction among tautologies. The authors of An and Arm [1] have provided a different picture of equality for classical tautologies. The notion of "determinative conjunct" is introduced in Chubaryan [2] on the basis of which the notion of strong equality of classical tautologies was suggested in An and Arm [1]. The idea to revise the notion of equivalence between tautologies in such way that is takes into account an appropriate measure of their "complexity".

The relations between the proof complexities of strongly equal classical tautologies in some proof systems are investigated in [3-5]. It was proved that the strongly equal tautologies have the same proof complexities in some "weak" proof systems, but the measures of proof complexities for strongly equal tautologies can be essentially different in the most traditional proof systems of Classical Logic (Frege systems, substitution Frege systems, sequent systems with and without cut rule). As the set of classical tautologies is co-NP-complete, the theory of proof complexity for classical proof systems is motivated by the conjecture NP $\neq$ co-NP. The set of tautologies, being intuinistically valid is PSPACE-complete, thus the PSPACE $\neq$ NP conjecture motivates an analogous research program as in classical case. In this work we introduce the notions of strongly equal non-classical valid sequents (non-classical tautologies) and show that the proof complexities of strongly equal non-classical valid sequents (non-classical tautologies) can be also essentially different in some sequent propositional systems of Intuitionistic, Johansson's and Monotone logics, therefore in corresponding Frege systems as well..

The second theme of our investigation is connected with relation between the proof complexities of minimal tautologies, i.e. tautologies, which are not a substitution of a shorter tautology, and results of a substitution in them. The minimal tautologies play main role in proof complexity area. Really all "hard" propositional formulaes, proof complexities of which are investigated in many well known papers, are minimal tautologies. There is traditional assumption that minimal tautology must be no harder than any substitution in it. We introduce for the propositional proof systems the notions of monotonous by lines and monotonous by sizes of proofs. In [4, 6] it is proved that many traditional classical proof systems of 2-valued and many-valued logics are no monotonous neither by lines nor by size. Here we prove the analogous result for some systems of non-classical propositional logic as well.

This work consists from 4 main sections. After Introduction we give the main notion and notations as well as some auxiliary statements in Preliminaries. The main results are given in 3-th section and in the last section we give some problem for discussion.

2. PRELIMINARIES

We will use the current concepts of a propositional formula, a classical tautology and non-classical tautologies, sequent, sequent systems for non-classical propositional logics [7-9] Frege systems for Intuitionistic and Johansson's logics [10, 11] and proof complexity [12]. Let us recall some of them.

2.1. The Considered Sequent Systems

Follow Kleene [7] we give the definition of main systems, which are considered in this point. The particular choice of a language for presented propositional formulas is immaterial in this consideration. However, because of some technical reasons we assume that the language contains the propositional variables p, q and p_i, q_i $(i \geq 1)$, logical connectives $\neg$, $\supset$, $\vee$, $\wedge$ and parentheses (,). Note that some parentheses can be omitted in generally accepted cases.

2.1.1. Sequent system uses the denotation of sequent $\Gamma \to \Delta$ where Γ (antecedent) and Δ (succedent) are finite (may be empty) sequences of propositional formulas.

For every formula C and for any sequence of formulas Γ the axiom scheme of propositional intuitionistic (***PI***) system is $C, \Gamma \to C$.

For every formulas A, B, for any sequence of formulas Γ and sequence Δ, which is empty or consists of one formula, the logic rules are.

$$\supset\to \frac{A \supset B,\ \Gamma \to \Delta,\ A \quad B,\ \Gamma \to \Delta}{A \supset B,\ \Gamma \to \Delta} \qquad\qquad \to\supset \frac{A,\ \Gamma \to B}{\Gamma \to A \supset B}$$

$$\vee\to \frac{A,\ A \vee B,\ \Gamma \to \Delta \text{ and } B,\ A \vee B,\ \Gamma \to \Delta}{A \vee B,\ \Gamma \to \Delta} \qquad\qquad \to\vee \frac{\Gamma \to A \text{ or } \Gamma \to B}{\Gamma \to A \vee B}$$

$$\wedge\rightarrow \frac{A,\ A\wedge B,\ \Gamma\rightarrow\Delta \text{ or } B,\ A\wedge B,\ \Gamma\rightarrow\Delta}{A\wedge B,\ \Gamma\rightarrow\Delta} \qquad \rightarrow\wedge \frac{\Gamma\rightarrow\ A \text{ and } \Gamma\rightarrow\ B}{\Gamma\rightarrow\ A\wedge B}$$

$$\neg\rightarrow \frac{\neg A,\ \Gamma\rightarrow\ A}{\neg A,\ \Gamma\rightarrow\Delta} \qquad \rightarrow\neg \frac{A,\ \Gamma\rightarrow}{\Gamma\rightarrow\ \neg A}$$

For propositional Johansson's (minimal) system (***PM***) axiom sxeme and inference rules are the same, but Δ must be empty [8]. Note that the order of formula ocurences in antecedents (succedents) are immaterial in above systems.

The propositional monotone system (***PMon***), where only monotonous logical functions are used for construction of formulas, we define follow Atserias, et al. [9].

The axioms of ***PMon*** system are

$$A\rightarrow A, \qquad \perp\rightarrow\Gamma, \qquad \Gamma\rightarrow T,$$

where A is any formula, Γ is sequence of formulas, by $\perp$ and T are denoted "false" and "truth" accordingly

For every formulas A , B and for any sequence of formulas Γ, Γ', Δ and Δ' the inference rules are.

$$(L_1)\frac{\Gamma,A,A,\Delta\rightarrow\Gamma'}{\Gamma,A,\Delta\rightarrow\Gamma'} \qquad (L_2)\frac{\Gamma,A,B,\Delta\rightarrow\Gamma'}{\Gamma,B,A,\Delta\rightarrow\Gamma'} \qquad (L_3)\frac{\Gamma\rightarrow\Gamma'}{\Gamma,A\rightarrow\Gamma'}$$

$$(L_4)\frac{A,B,\Gamma\rightarrow\Delta}{A\wedge B,\Gamma\rightarrow\Delta} \qquad (L_5)\frac{A,\Gamma\rightarrow\Delta \text{ and } B,\Gamma'\rightarrow\Delta'}{A\vee B,\Gamma,\Gamma'\rightarrow\Delta,\Delta'}$$

$$(R_1)\frac{\Gamma'\rightarrow\Gamma,A,A,\Delta}{\Gamma'\rightarrow\Gamma,A,\Delta} \qquad (R_2)\frac{\Gamma'\rightarrow\Gamma,A,B,\Delta}{\Gamma'\rightarrow\Gamma,B,A,\Delta} \qquad (R_3)\frac{\Gamma'\rightarrow\Gamma}{\Gamma'\rightarrow\Gamma,A}$$

$$(R_4)\frac{\Gamma\rightarrow\Delta,A,B}{\Gamma\rightarrow\Delta,A\vee B} \qquad (R_5)\frac{\Gamma\rightarrow\Delta,A \text{ and } \Gamma'\rightarrow\Delta',B}{\Gamma,\Gamma'\rightarrow\Delta,\Delta',A\wedge B}$$

To all above systems can be added cut-rule of inference

$$\frac{\Gamma\rightarrow\Delta,A \quad A,\Gamma'\rightarrow\Delta'}{\Gamma,\Gamma'\rightarrow\Delta,\Delta'}.$$

We use the well known notion of proof in all above systems.

Any sequent $\Gamma\rightarrow\Delta$ is called I-valid sequent (M-valid sequent, Mon-valid sequent) if it is deduced in the system ***PI (PM, PMon)***. Any formula A is called I-tautology or M-tautology if sequent $\rightarrow A$ is deduced in the corresponding system ***PI*** or ***PM.*** Any formula A⊃B is called Mon-tautology if sequent $A\rightarrow$B is deduced in the system ***PMon.***

Remark 2.1.1

1) Every Mon-tautology (Mon-valid sequent) is M-tautology (M-valid sequent), every M-tautology (M-valid sequent) is I-tautology (I-valid sequent) and every I-tautology (I-valid sequent) is classical tautology (classical valid sequent).

2) If any formula is not classical tautology, then it is not non-classical tautology as well.

Let $\Gamma \rightarrow \Delta$ be some sequent, where Γ is a sequence of formulas $A_1, A_2, \dots, A_l$ $(l \geq 0)$ and Δ is a sequence of formulas $B_1, B_2, \dots, B_m$ $(m \geq 0)$. The formula form of sequent (f.f.s.) $\Gamma \rightarrow \Delta$ is the formula $\varphi_{\Gamma\rightarrow\Delta}$, which is defined usually as follows:

1) $A_1 \wedge A_2 \wedge \dots \wedge A_l \supset B_1 \vee B_2 \vee \dots \vee B_m \quad l, m \geq 1$,

2) $A_1 \wedge A_2 \wedge \dots \wedge A_l \supset \perp$ for $l \geq 1$, $m = 0$,

3) $B_1 \vee B_2 \vee \dots \vee B_m$ for $l = 0, m \geq 1$.

It is well-known that $\Gamma \rightarrow \Delta$ is ***classical (intuitionistic, Johansson's, monotone) valid sequent iff (if and only if) its f.f.s. is classical (intuitionistic, Johansson's, monotone) tautology.***

For every inference rule $\frac{D}{E}$ ($\frac{D \text{ and } D'}{E}$) we call ***inference formula form (i.f.f.)*** the formula *f.f.s.D⊃f.f.s.E* (*f.f.s.D⊃(f.f.s.*D'*⊃f.f.s.E*)).

Sometimes we'll use term *tautology* (*valid sequent*) for all types of above mentioned tautology (valid sequent) further.

2.2. Some Properties of Tautologies (Valid Sequents)

Here we give some properties of propositional formulas, which will be used for main results.

2.2.1. Determinative Disjunctive Normal Forms

Following the usual terminology we call the variables and negated variables literals for classical logic. The conjunct K (clause) can be represented simply as a set of literals (no conjunct contains a variable and its negation simultaneously).

In [1, 2] the following notions were introduced for classical logic. Each of the under-mentioned trivial identities for a propositional formula ψ is called *replacement-rule:*

$0\&\psi = 0, \quad \psi\&0 = 0, \quad 1\&\psi = \psi, \quad \psi\&1 = \psi,$

$0 \vee \psi = \psi, \quad \psi \vee 0 = \psi, \quad 1 \vee \psi = 1, \quad \psi \vee 1 = 1,$

$0 \supset \psi = 1, \quad \psi \supset 0 = \neg\psi, \quad 1 \supset \psi = \psi, \quad \psi \supset 1 = 1,$

$\neg 0 = 1, \quad \neg 1 = 0, \quad \neg\neg\psi = \psi.$

Application of a replacement-rule to some word consists in the replacing of some its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

Let φ be a propositional formula, $P=\{p_1,p_2,\dots,p_n\}$ be the set of all variables of φ, and $P'=\{p_{i_1},p_{i_2},\dots,p_{i_m}\}$ $(1\le m\le n)$ be some subset of P.

Definition 2.2.1.1

Given $\sigma=\{\sigma_1,\sigma_2,\dots,\sigma_m\}\subset E^m$, the conjunct $K^\sigma=\{p_{i_1}{}^{\sigma_1},p_{i_2}{}^{\sigma_2},\dots,p_{i_m}{}^{\sigma_m}\}$ is called $\varphi-1$ -determinative ($\varphi-0$ -determinative) if assigning $\sigma_j(1\le j\le m)$ to each p_{ij} and successively using replacement-rules we obtain the value of φ (1 or 0) independently of the values of the remaining variables.

Definition 2.2.1.2

DNF $D=\{K_1,K_2,\dots,K_j\}$ is called determinative DNF (DDNF) for φ if $\varphi=D$ and every conjunct K_i $(1\le i\le j)$ is 1-determinative for φ.

Definition 2.2.1.3

DNF $D=\{K_1,K_2,\dots,K_j\}$ is called determinative DNF (dDNF) for φ if $\varphi=D$ and every conjunct K_i $(1\le i\le j)$ is 1-determinative for φ.

It is obvious, that for every classical tautology each corresponding dDNF must be also classical tautology. Some arguments for the following definition were given in An and Arm [1].

The classical tautologies φ and ψ are strongly equal if every determinative conjunct for φ is determinative conjunct for ψ and vice versa.

It is not difficult to see, that dDNF for classical tautology can be constructed directly. As the non-classical validity is determined by derivability in some accordingly propositional proof system, the above definition of dDNF for non-classical tautologies is not applicable. In Chubaryan [2] some algorithm for construction of dDNF for classical tautologies on the base of their resolution refutations was given. The analogies of dDNF for intuitionistic and Johansson's tautologies ($\varphi-$ I-determinative DNF and $\varphi-$ M-determinative DNF accordingly) were constructed on the base of proofs in intuitionistic and minimal resolution systems [13] where, in particularly, were showed, that ***only variables with one or double negations are the literals in I-determinative conjuncts and $p\supset\perp$ or $(p\supset\perp)\supset\perp$ type formulas are literals in M-determinative conjuncts.*** The Mon-determinative DNF for every Mon-tautology $A\supset B$ can by constructed by analogy on the base of some ***PMon-***proof of sequent

$A \rightarrow B$. As literals for *monotone* logic can be p^+ and p^-, depending of positive or negative occurrence of variable p in the axioms of ***PMon***-proof of sequent $A \rightarrow B$.

Definition 2.2.1.4

DNF is called dDNF for a valid sequent if it is dDNF for its f.f.s..

Main definition 1. The classical (intuitionistic, Johansson's) tautologies φ *and* ψ *are strongly equal if every dDNF (I-dDNF, M-dDNF) for* φ *is dDNF (I-dDNF, M-dDNF) for* ψ *and vice versa. The classical (intuitionistic, Johansson's, monotone) valid sequents* $\Gamma \rightarrow \Delta$ *and* $\Gamma' \rightarrow \Delta'$ *are strongly equal if every dDNF (I-dDNF, M-dDNF, Mon-dDNF) for* $\Gamma \rightarrow \Delta$ *is dDNF (I-dDNF, M-dDNF, Mon-dDNF) for* $\Gamma' \rightarrow \Delta'$ *and vice versa.*

2.3. Essential Subformulas of Tautologies (Valid Sequents)

For proving the main results we generalize for non-classical tautologies the notion of *essential subformulas*, introduced in Chubaryan [2].

Let F be some formula and $Sf(F)$ be the set of all non-elementary subformulas of formula F . For every formula F , for every $\varphi \in Sf(F)$ and for every variable p by F_{φ}^{p} is denoted the result of the replacement of the subformulas φ everywhere in F by the variable p . If $\varphi \notin Sf(F)$, then F_{φ}^{p} is F .

We denote by $Var(F)$ the set of all variables in F .

Definition 2.3.1

Let p be some variable that $p \notin Var(F)$ and $\varphi \in Sf(F)$ for some classical tautology (I-tautology, M-tautology, Mon-tautology) F . We say that φ is an ***essential subformula*** in F iff F_{φ}^{p} is ***not classical*** tautology.

Note that for example the subformula $\neg\neg A$ is not essential for I-tautology $\neg\neg\neg A \vee \neg\neg A$, because the formula $\neg p \vee p$ is not I-tautology, but is classical tautology.

The set of essential subformulas in tautology F we denote by *Essf(F)*, the number of essential subformulas – by *Nessf(F)* and the sum of sizes of all essential subformulas by *Sessf(F)*.

Definition 2.3.2

A tautolgy is called ***minimal*** if it is not a substitution of a shorter tautology.

Definition 2.3.3

Sequent $\Gamma \rightarrow \Delta$ is called **minimal valid** if its formula form $\varphi_{\Gamma\rightarrow\Delta}$ is minimal tautology.

We denote by S(φ) the set of all formulas, every of which is result of some substitution in a minimal tautology φ.

If F is minimal tautology, then $Essf(F) = Sf(F)$.

Definition 2.3.3

The subformula φ is essential for valid sequent $\Gamma \rightarrow \Delta$ if it is essential for its formula form.

It easy to prove the following statements.

Proposition 2.3

Let $\mathcal{F}$ be some of above proof system (with and without cut rule), F be a valid sequent and $\varphi \in Essf(F)$, then

- in every $\mathcal{F}$-proof of F subformula φ must be essential either at least in some axiom, used in proof or in i.f.f. for some used in proof inference rule,
- there is some constant c such that the number of essential subformulas for every axiom of $\mathcal{F}$ and of i.f.f. for every inference rule of $\mathcal{F}$ is no more, than c.

Both statements of this Proposition can be proved by immediate examination every of axioms and inference rules in each of above systems. The analogous statements for traditional proof systems of classical systems are proved in Chubaryan [2].

2.4. Proof Complexity Measures

By $|\varphi|$ we denote the size of a formula φ, defined as the number of all logical signs in it. It is obvious that the full size of a formula, which is understood to be the number of all symbols is bounded by some linear function in $|\varphi|$.

In the theory of proof complexity two main characteristics of the proof are: ***t*- *complexity*** (length), defined as the number of proof steps, ***l*-*complexity*** (size), defined as sum of sizes for all formulas in proof [12].

Let Φ be a proof system and $\Gamma \rightarrow \Delta$ be a valid sequent. We denote by $t^{\Phi}_{\Gamma\rightarrow\Delta}$ ($l^{\Phi}_{\Gamma\rightarrow\Delta}$) the minimal possible value of *t*-complexity (l-complexity) for all Φ-proofs of $\Gamma \rightarrow \Delta$.

Main Definition 2. *Sequent proof system* Φ *is called t-monotonous (*l*-monotonous) if for every valid sequent* $\Gamma \rightarrow \Delta$ *and for every sequent* $\Gamma_1 \rightarrow \Delta_1$ *such that* $\varphi_{\Gamma_1 \rightarrow \Delta_1} \in S(\varphi_{\Gamma \rightarrow \Delta})$ $t^{\phi}_{\Gamma \rightarrow \Delta} \leq t^{\phi}_{\Gamma_1 \rightarrow \Delta_1}$ $(l^{\phi}_{\Gamma \rightarrow \Delta} \leq l^{\phi}_{\Gamma_1 \rightarrow \Delta_1})$.

3. MAIN RESULTS

3.1. Auxiliary Statements

Before we prove the main theorems, at first we must give some easy proved auxiliary statements. Let us consider the following sequences of sequents:

$$D_n = p \rightarrow \overbrace{p \vee (p \vee (p \vee \dots \vee (p \vee p) \dots))}^{n},$$

$$E_n = p \rightarrow \overbrace{p \wedge (p \wedge (p \wedge \dots \wedge (p \wedge p) \dots))}^{n},$$

$$F_n = p \rightarrow q \vee \overbrace{p \wedge (p \wedge (p \wedge \dots \wedge (p \wedge p) \dots))}^{n},$$

$$G_n = p \rightarrow (p \wedge p) \vee \overbrace{p \wedge (p \wedge (p \wedge \dots \wedge (p \wedge p) \dots))}^{n}.$$

Lemma 3.1.a) There are constants c_1, c_2, c_3 and c_4 such, that for every n

$$t^{PMon}_{D_n} \leq c_1, \quad t^{PMon}_{G_n} \leq c_2 \text{ and } l^{PMon}_{D_n} \leq c_3 n, \quad l^{PMon}_{G_n} \leq c_4 n.$$

b) There are constants k_1, k_2, k_3 and k_4 such, that for every n

$$t^{PI}_{E_n} \geq k_1 n, \quad t^{PI}_{F_n} \geq k_2 n \text{ and } l^{PI}_{E_n} \geq k_3 n^2, \quad l^{PI}_{F_n} \geq k_4 n^2.$$

Proof of point a) is obviously. Really for every n sequent D_n can be proved in ***PMon*** as follow

$$\cfrac{\cfrac{p \rightarrow p}{p \rightarrow p, \overbrace{p \vee (p \vee (p \vee \dots \vee (p \vee p) \dots))}^{n-1}}}{Dn}.$$

For every n sequent G_n can be proved in ***PMon*** as follow

$$\cfrac{\cfrac{\cfrac{p \rightarrow p \text{ and } p \rightarrow p}{p \rightarrow p \wedge p}}{p \rightarrow p \wedge p, \overbrace{p \wedge (p \wedge (p \wedge \dots \wedge (p \wedge p) \dots))}^{n}}}{Gn}$$

For proving of point b) note that for each i $(1\leq i\leq n)$ the formula $\overbrace{p \wedge (p \wedge \ldots \wedge (p \wedge p) \ldots))}^{i}$ is essential both for E_n and F_n, therefore $Nessf(E_n)\geq n$, $Nessf(F_n)\geq n$ and $Sessf(E_n) \geq n^2/2$, $Sessf(E_n) \geq n^2/2$. Now we must use the statements of both points from Propositional 2.3. □

Corollary. Above statements for sequents D_n and E_n are true in the systems ***PI*** and ***PM*** as well, Therefore above statements for sequents G_n and F_n are true in the systems ***PM*** and ***PMon*** as well.

Theorem 1. a) The ***intuitionistic, Johansson's*** and ***monotone valid*** seguents $\boldsymbol{D_n}$ and $\boldsymbol{E_n}$ are strongly equal.

b) For every of above mentioned system $\mathcal{F}$ (with and without cut rule)

$t^{\mathcal{F}}_{D_n} = O(1)$ and $l^{\mathcal{F}}_{D_n} = O(n)$, but $t^{\mathcal{F}}_{E_n} = \Omega(n)$ and $l^{\mathcal{F}}_{E_n} = \Omega(n^2)$.

Proof. It is not difficult to see that ***I-dDNF*** of $\boldsymbol{D_n}$ and $\boldsymbol{E_n}$ is $\{\neg p, \neg\neg p\}$, ***M-dDNF*** of $\boldsymbol{D_n}$ and $\boldsymbol{E_n}$ is $\{p \supset \perp, (p \supset \perp) \supset \perp\}$, and ***Mon-dDNF*** of $\boldsymbol{D_n}$ and $\boldsymbol{E_n}$ is $\{p^+, p^-\}$, therefore seguents $\boldsymbol{D_n}$ and $\boldsymbol{E_n}$ are strongly equal.

Proof of point b) follows from Corollary of Lemma 3.1.. □

Theorem 2. Every of above mentioned systems $\mathcal{F}$ (with and without cut rule) is neither *t-monotonous* nor *l-monotonous.*

Proof. It is not difficult to see that for every n sequent F_n is minimal valid sequent and corresponding sequent G_n is result of substitution in F_n. From Corollary of Lemma 3.1. it is follow that $t^{\mathcal{F}}_{G_n} = O(1)$ and $l^{\mathcal{F}}_{G_n} = O(n)$, but

$t^{\mathcal{F}}_{F_n} = \Omega(n)$ and $l^{\mathcal{F}}_{F_n} = \Omega(n^2)$. □

3.2. Results for Frege Systems of Intuitionistic and Johansson's Logics

Here we recall the definitions of Frege systems for Intuitionistic and Johansson's logics, which are given in Mints and Kozhevnikov [10] and Sayadyan and Chubaryan [11] correspondingly.

Definition 3.2.1

Some inference rule $\frac{A_1,\ A_2,\ \ldots,\ A_n}{B}$ is called *admissible* for some Hilbert style proof system Φ if formula B can be deduced in this system from the premises $A_1,\ A_2,\ \ldots,\ A_n$.

Let $\mathbf{I}_1$ and $\mathbf{M}_1$ are the following systems (see, for example Kleene [7]).

For each propositional formulas A, B, C every from the following formula is axioms scheme of $\mathbf{I}_1$

1) $A \supset (B \supset A)$

2) $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$

3) $A \supset (B \supset A \wedge B)$

4) $A \wedge B \supset A$; $A \wedge B \supset B$

5) $A \supset A \vee B$; $B \supset A \vee B$

6) $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

7) $(A \supset B) \supset ((A \supset \neg B) \supset \neg A)$

8) $\neg A \supset (A \supset B)$

Inference rule is *modus ponens* $\frac{A, A \supset B}{B}$ (m.p.).

The system $\mathbf{M}_1$ has only axioms scheme 1)-7) and instead if negation $\neg A$ is used $A \supset \perp$.

For definition of Frege systems for Intuitionistic and Johansson's logics we use the generally accepted notion of polynomial simulation. Let Φ_1 and Φ_2 be two different proof systems.

Definition 3.2.2

The system Φ_1 *p-simulates* the system Φ_2 if there exist the polynomial *p()* such, that for each formula φ

provable both in the systems Φ_1 and Φ_2, we have $l_{\varphi}^{\Phi 1} \leq p(l_{\varphi}^{\Phi 2})$.

Definition 3.2.3

The systems Φ_1 and Φ_2 are *p-equivalent*, if systems Φ_1 and Φ_2 *p-l-simulate* each other.

Definition 3.2.4

Every Frege system for Intuitionistic (Johansson's) logic ***FI*** (***FM***) consists from finite set of axioms schemes, each of which is provable in $\mathbf{I}_1$ ($\mathbf{M}_1$), finite set of inference rules, each of which is admissible in $\mathbf{I}_1$ ($\mathbf{M}_1$), and ***FI*** (***FM***) is *p*-equivalent with $\mathbf{I}_1$ ($\mathbf{M}_1$).

It is not difficult to prove, that the statements of the Theorems 1. and 2.are valid for Frege systems of Intuitionistic and Johansson's logics, using the formulas:

$$(D)'_n = p \supset \overbrace{p \vee (p \vee (p \vee \dots \vee (p \vee p) \dots))}^{2^n},$$

$$(E)'_n = p \supset \overbrace{p \wedge (p \wedge (p \wedge \dots \wedge (p \wedge p) \dots))}^{2^n},$$

$$(F)'_n = \; p \supset q \vee \overbrace{p \wedge (p \wedge (p \wedge \ldots \wedge (p \wedge p) \ldots))}^{2^n},$$

$$(G)''_n = \; p \supset (p \wedge p) \vee \overbrace{p \wedge (p \wedge (p \wedge \ldots \wedge (p \wedge p) \ldots))}^{2^n}.$$

4. DISCUSSION

We want to note, that for every n the sequent G_n is result of substitution in the other minimal valid sequent $p \to (p \wedge p) \vee q$, t-complexity and l-complexity of which is bounded by some constant in all above mentioned systems. We can introduce the following definition: ***the sequent proof system Φ is called t-strongly monotonous (l -strongly monotonous) if for every valid sequent $\Gamma \to \Delta$ there is minimal valid sequent $\Gamma_1 \to \Delta_1$ such that $\varphi_{\Gamma\to\Delta} \in S(\varphi_{\Gamma_1\to\Delta_1})$ and $t^{\Phi}_{\Gamma_1\to\Delta_1} \leq t^{\Phi}_{\Gamma\to\Delta}$ ($l^{\Phi}_{\Gamma_1\to\Delta_1} \leq l^{\Phi}_{\Gamma\to\Delta}$).*** It is interesting to investigate the following problem: are the above non-classical systems as well as the classical systems strongly monotonous? It seems that answer must be positive. Analogous question for tree like proofs was stated in Anikeev [14]. Investigation of this questions are in process.

REFERENCES

[1] C. An and C. Arm, "Definition of strong equality of tautologies and universal system for various propositional logics," *Armenian Journal of Mathematics*, vol. 1, pp. 30-35, 2008. *View at Google Scholar*

[2] A. A. Chubaryan, "On complexity of the proofs in Frege system," presented at the CSIT Conference, Yerevan, 2001

[3] C. An, C. Arm, and A. Mnatsakanyan, "Proof complexities of strongly equal classical tautologies in some proof systems," *Nauka i Studia, NR*, vol. 42, pp. 92-98, 2013.

[4] C. Anahit and P. Garik, "Frege systems are no monotonous, evolutio," *Natural Sciences, Vyhn*, vol. 3, pp. 12-14, 2016.

[5] C. Anahit and P. Garik, "On some properties of several proof systems for 2-valued and 3-valued propositional logic," *Fundamentalis Scientiam, Spain*, vol. 8, pp. 70-73, 2017.

[6] C. Anahit, N. Hakob, K. Arman, and P. Garik, "Propositional sequent systems of two valued classical logic and many valued logics are no monotonous," *ASL, ESM, Logic Colloquium – 2017, Stockholm, Abstracts, Programme and Abstracts, LC2017-Book, 98*, 2017.

[7] S. C. Kleene, *Introduction to Metamathemics*: D.Van Nostrand Company, INC, 1952.

[8] A. Chubaryan and O. Bolibekyan, "On sequential systems of weak arithmetics," *DNA of Armenia, Applied Mathematics*, vol. 102, pp. 214-218, 2002.

[9] A. Atserias, N. Galesi, and R. Gavalda, "Monotone proofs of the pigeon hole principle," *Mathematical Logic Quarterly*, vol. 47, pp. 461-474, 2001. *View at Google Scholar | View at Publisher*

[10] G. Mints and A. Kozhevnikov, "Intuitionistic Frege systems are polynomially equivalent," *Zapiski Nauchnykh Seminarov POMI*, vol. 316, pp. 129-146, 2004.

[11] S. Sayadyan and A. Chubaryan, *On polynomially equivalence of minimal Frege systems* vol. 27. Yerevan: Mathematical Problemsof Computer Science, 2007.

[12] Y. Filmus, M. Lauria, J. Nordstrom, N. Thapen, and N. Ron-Zewi, "Space complexity in polynomial calculus," presented at the 2012 IEEE Conference on Computational Complexity (CCC), 2012.

[13] A. Chubaryan, A. Chubaryan, H. Nalbandyn, and S. Sayadyan, "A hierarchy of resolution systems with restricted substituted rules," *Computer Technology and Applications*, vol. 3, pp. 330-336, 2012. View at Google Scholar

[14] A. S. Anikeev, "On some classification of deducible propositional formulas," *Matematicheskie Zametki*, vol. 11, pp. 165-174, 1972. View at Google Scholar