Journal of Asian Scientific Research
ISSN(e): 2223-1331
ISSN(p): 2226-5724
DOI: 10.18488/journal.2.2018.810.287.292
Vol. 8, No. 10, 287-292

URL: www.aessweb.com

ON APPROXIMATE SOLUTIONS FOR TIME-FRACTIONAL DIFFUSION EQUATION

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Article History
Received: 23 August 2018
Revised: 28 September 2018
Accepted: 5 November 2018
Published: 30 November 2018

Keywords
Preconditioned rotated method
Time-fractional diffusion
Equation

ABSTRACT

In the last decades differential equations involving fractional derivatives and integrals have been studied by many researchers. Due to their ability to model more adequately some phenomena, fractional partial differential equations have been used in numerous areas such as finance, hydrology, porous media, engineering and control systems, etc. Numerical schemes based on rotated finite difference approximation have been proven to work well in solving standard diffusion equations. However, the formulation of these strategies on time fractional diffusion counterpart is still at its infancy. A well-designed preconditioning for these types of problems reduces the number of iterations to reach convergence. In this research work, we have derived new preconditioned fractional rotated finite difference method for solving 2D time-fractional diffusion equation. Numerical experiments are conducted to examine the effectiveness of the proposed method.

Contribution/ Originality: This study contributes in the existing literature about the foundation of fast iterative schemes from the preconditioned methods for solving the time-fractional diffusion equation. It is one of the few studies which combine a suitable pre-conditioner matrix with the rotated iterative scheme as a way to further improve the convergence rate of the method in solving the 2D time-fractional diffusion equation.

1. INTRODUCTION

The importance of this study lies in the various applications of fractional partial differential equations (FPDE's) in finance, physics, image processing and engineering [1, 2]. It is well known that FPDE's is a generalized of the classical partial differential equations (PDE's). As a result of that there is no general method that can be used in solving FPDE's same as classical PDE's. Approximation methods such as finite difference methods have played important role for solving FPDE's in the last few years [3, 4]. It is noteworthy to observe that the finite difference schemes derived from skewed (rotated) difference operators have been extensively investigated over the years for solving FPDE's. These iterative methods have been shown to be much faster than the methods based on the standard five-point formula which is due to the formers' overall lower computational complexities (Saeed and Ali [5]; Ali and Saeed [6]; Saeed and Ali [7]. In Saeed [8] the preconditioned rotated finite difference method applied successfully for solving fractional elliptic partial differential equations and the reveal results was very encouraging.

This work involves an investigation on the utilization of the new preconditioned fractional rotated finite difference method for solving 2D Time-Fractional Diffusion Equations. An outline of this paper is as follows. In Section 2, the proposed accelerated version of fractional rotated five point's approximation method will be formulated. The numerical results will be presented to show the efficiency of the new proposed methods in Section 3. Finally, Conclusion and Future work are given in Section 4.

2. FORMULATION OF THE PROPOSED PRECONDITIONED ITERATIVE METHOD

Consider the following time fractional diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+f(x,y,t), \tag{1}$$

where α is the order of the time fractional derivative in Caputo sense which is defined as Zhang and Sun [9].

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,\xi)}{\partial \xi}\frac{d\xi}{(t-\xi)^\alpha}\,,\ 0<\alpha<1. \tag{2}$$

Suppose that the domains are constant for both x and y, while the grid dimensions in relation to space and time for the positive integers n and l are respectively represented by $h=\frac{1}{n}$ and $\tau=\frac{T}{l}$. The grid points in the space interval $[0,1]$ are denoted $x_i=ih,\ x_j=jh,\ \{i,j=0,1,\ldots,n\}$ and the grid points for time are designated $t_k=k\tau,\ k=0,1,\ldots,l.$ Discretization with regard to time fractional with utilization of Crank-Nicolson finite difference approximations at $(x_i,y_j,t_{n+1/2})$ is realized through the formula displayed below [10]

$$\frac{\partial^\alpha u(x_i,y_j,t_{k+1})}{\partial t^\alpha}=\{w_1 u^k+\sum_{s=1}^{k-1}[w_{k-s+1}-w_{k-s}]u_{i,j}^s-w_k u_{i,j}^0+\sigma\frac{(u_{i,j}^{k+1}-u_{i,j}^k)}{2^{1-\alpha}}\}+0(\tau^{2-\alpha}), \tag{3}$$

where $\sigma=\frac{1}{\tau^\alpha\Gamma(2-\alpha)}$, $w_s=\sigma\{(\frac{s+1}{2})^{1-\alpha}-(\frac{s-1}{2})^{1-\alpha}\}$

utilization of the standard second order Crank-Nicolson difference scheme with the formula (3) for finite difference discretization of (1) will result in the standard Crank-Nicolson formula portrayed below

$$u_{i,j}^{k+1}=\frac{1}{1+2r}\{\frac{r}{2}[(u_{i+1,j}^{k+1}+u_{i-1,j}^{k+1}+u_{i,j+1}^{k+1}+u_{i,j-1}^{k+1})+(u_{i+1,j}^{k}+u_{i-1,j}^{k}+u_{i,j+1}^{k}+u_{i,j-1}^{k})]+(1-2^{1-\alpha}w_1^*-2r)u_{i,j}^k+2^{1-\alpha}\sum_{s=1}^{k-1}[(w_{k-s}^*-w_{k-s+1}^*)u_{i,j}^s+2^{1-\alpha}\ w_k^*u_{i,j}^0+m_0\ f_{i,j}^{\frac{k+1}{2}}\}, \tag{4}$$

where $m_0=\tau^\alpha\Gamma(2-\alpha)*2^{1-\alpha}$, $r=\frac{m_0}{h^2}$, $w_s^*=[(\frac{s+1}{2})^{1-\alpha}-(\frac{s-1}{2})^{1-\alpha}]$.

It can be observed that for the rotated five-point finite difference approximation the following transformations take place

$$i\,,j\pm1\rightarrow i\pm1,j\pm1$$
$$i\pm1,j\rightarrow i\pm1,j\mp1$$
$$h\rightarrow\sqrt{2}h.$$

Therefore, this approximation method (achieved through 45^0 degree clockwise rotation of the x-y axis) for equation (1) can be written as the following:

$$u_{i,j}^{k+1}=\frac{1}{1+r}\{\frac{r}{4}[(u_{i+1,j+1}^{k+1}+u_{i-1,j-1}^{k+1}+u_{i-1,j+1}^{k+1}+u_{i+1,j-1}^{k+1})+(u_{i+1,j+1}^{k}+u_{i-1,j-1}^{k}+u_{i-1,j+1}^{k}+u_{i+1,j-1}^{k})]$$
$$+(1-2^{1-\alpha}w_1^*-r)u_{i,j}^k+2^{1-\alpha}\sum_{s=1}^{k-1}(w_{k-s}^*-w_{k-s+1}^*)u_{i,j}^s+2^{1-\alpha}\ w_k^*u_{i,j}^0+m_0\ f_{i,j}^{\frac{k+1}{2}}, \quad (5)$$

wher m_0, r and w_s^* as mentioned before in (4).

Several preconditioned strategies have been used for improving the convergence rate of the iterative methods derived from the standard and skewed (rotated) finite difference operators [11-14]. The difficulty lays in construct the suitable preconditioners which transform the resulted system of iterative method to new preconditioned system. This new preconditioned system has same exact solution but has more favorable spectral properties.

Usually the resulted system from equation (5) is large and the coefficient matrix A is sparse. Therefore, matrix A can be write as

$$A=D-L-U \quad (6)$$

where D is diagonal matrix A, $-L$ is strictly lower triangular part of A and $-U$ is strictly upper triangular part of A.

In Saeed [15] the preconditioned fractional rotated finite difference method was successfully applied to solve 2D time-fractional diffusion equation by using the preconditioned matrix $P_1=(I+kU)$ which modify the original system resulted from fractional rotated finite difference method into new system that is equivalent in the sense that it has the same solution, but that has more favorable spectral properties. However, the iteration count for the preconditioned system decreased about only 23-30% compared to the original system.

Inspired by the works above, we derive new preconditioned fractional rotated finite difference system by using the preconditioned matrix $P_2=(I-kL)$ where $0\le k<1.5$. In the following section, we will show that the iteration count for the proposed preconditioned system decreased about 30-50% compared to the original system which yield very encouraging results.

3. NUMERICAL RESULTS

In this section, we present numerical results for the proposed method applied to two particular examples. The first problem as the following [16]:

$$\frac{\partial^\alpha u}{\partial t^\alpha}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+[\Gamma(2+\alpha)t-2t^{1+\alpha}]e^{x+y} \quad (7)$$

where the solution domain is $\Omega=\{(x,y):0\le x\le 1,\ 0\le y\le 1\}$, with Dirichlet boundary requistes which comply with the exact solution $u(x,y,t)=e^{x+y}\,t^{1+\alpha}$. Ultimately, a Gauss-Sidel method holding a relaxation value equivalent to 1 was applied on a variety of grid dimensions (4, 8,16, 20 and 24) with varying time steps $(\frac{1}{4},\frac{1}{8},\frac{1}{16},\frac{1}{20},\frac{1}{24})$ for $0<t<1$. Preconditioned methods were deemed efficient through investigations which revealed their superiority in the context of execution time (measured in seconds), number of iterations (Ite) and maximum absolute error (Max) with tolerance $\varepsilon=10^{-6}$.

Table-1. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.25$

Problem (1)					
Δt	n	*Method*	Time	iterations	Max Error
$\frac{1}{4}$	4	FRFD	0.0148	8	7.36E-3
		P_1FRFD	0.0139	7	7.35E-3
		P_2FRFD	0.0137	6	7.34E-3
$\frac{1}{8}$	8	FRFD	0.5910	26	2.13E-3
		P_1FRFD	0.5781	20	2.13E-3
		P_2FRFD	0.5660	18	2.11E-3
$\frac{1}{16}$	16	FRFD	4.8145	46	8.04E-3
		P_1FRFD	4.7324	38	8.21E-3
		P_2FRFD	4.5231	33	8.031E-3
$\frac{1}{20}$	20	FRFD	224.014	72	7.36E-4
		P_1FRFD	223.832	58	6.22E-4
		P_2FRFD	215.071	49	6.19E-4
$\frac{1}{24}$	24	FRFD	244.492	134	2.08E-4
		P_1FRFD	202.012	104	1.28E-4
		P_2FRFD	183.615	88	1.24E-4

Source: software results

Table-2. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.75$

Problem (1)					
Δt	n	*Method*	Time	iterations	Max Error
$\frac{1}{4}$	4	FRFD	0.0136	8	3.36E-3
		P_1FRFD	0.0136	7	3.33E-3
		P_2FRFD	0.0131	6	3.32E-3
$\frac{1}{8}$	8	FRFD	0. 5237	22	1.41E-3
		P_1FRFD	0.4301	18	1.48E-3
		P_2FRFD	0.4291	16	1.47E-3
$\frac{1}{16}$	16	FRFD	3.8211	34	5.22E-4
		P_1FRFD	3.1731	29	4.71E-4
		P_2FRFD	3.0211	26	4.68E-4
$\frac{1}{20}$	20	FRFD	128.231	72	3.76E-4
		P_1FRFD	102.038	51	2.61E-4
		P_2FRFD	100.121	48	2.57E-4
$\frac{1}{24}$	24	FRFD	142.091	84	3.15E-3
		P_1FRFD	123.512	67	2.12E-3
		P_2FRFD	111.217	53	2.08E-3

Source: software results

The attainment of a solution to an additional test problem (2) [9] substantiates the efficiency of these procedures:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \left(\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 2t^2\right)[\sin(x)\sin(y)], \quad 0 < \alpha < 1, \tag{8}$$

with initial and boundary conditions:

$$\begin{gathered} u(x,y,0) = 0, \quad u(0,y,t) = 0, \; u(x,0,t) = 0 \\ u(1,y,t) = t^2 \sin(1)\sin(y), \quad u(x,1,t) = t^2 \sin(x)\sin(1), \\ 0 < t < 1, \qquad\qquad 0 < x, y < 1. \end{gathered}$$

The exact solution is $u(x,y,t) = t^2 \sin(x)\sin(y)$.

Numerical data of the original FRFD and the preconditioned systems P_1FRDF & P_2FRDF are summarized in tables (1) , (2), (3) and (4) for problems (1) and (2) with $\alpha = 0.25$ and $\alpha = 0.75$ respectively.

Table-3. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.25$

Problem (2)					
Δt	n	Method	Time	iterations	Max Error
$\frac{1}{4}$	4	FRFD	0.0132	6	4.81E-4
		P_1FRFD	0.0132	6	4.81E-4
		P_2FRFD	0.0124	5	4.76E-4
$\frac{1}{8}$	8	FRFD	0.3682	21	1.22E
		P_1FRFD	0.3221	16	1.22E
		P_2FRFD	0.2972	12	1.18E
$\frac{1}{16}$	16	FRFD	3.1051	37	2.09E-5
		P_1FRFD	2.8626	23	2.38E-5
		P_2FRFD	2.4224	17	2.36E-5
$\frac{1}{20}$	20	FRFD	98.015	70	2.67E-5
		P_1FRFD	82.341	50	2.39E-5
		P_2FRFD	62.994	35	2.37E-5
$\frac{1}{24}$	24	FRFD	143.018	90	5.01E-5
		P_1FRFD	96.878	65	4.83E-5
		P_2FRFD	74.631	43	4.29E-5

Source: software results

Table-4. Comparison of the number of iterations, Execution time and maximum error for $\alpha = 0.75$

Problem (2)					
Δt	n	Method	Time	iterations	Max Error
$\frac{1}{4}$	4	FRFD	0.0135	7	1.98E-4
		P_1FRFD	0.0135	6	1.98E-4
		P_2FRFD	0.0102	5	1.83E-4
$\frac{1}{8}$	8	FRFD	0.3613	18	1.26E-4
		P_1FRFD	0.2851	14	1.14E-4
		P_2FRFD	0.2237	10	1.12E-4
$\frac{1}{16}$	16	FRFD	2.5064	33	1.05E-4
		P_1FRFD	2.0051	25	1.03E-4
		P_2FRFD	1.6628	18	1.01E-4
$\frac{1}{20}$	20	FRFD	87.411	52	1.31E-4
		P_1FRFD	66.725	38	1.28E-4
		P_2FRFD	42.725	22	1.26E-4
$\frac{1}{24}$	24	FRFD	117.318	74	1.82E-4
		P_1FRFD	98.702	58	1.61E-4
		P_2FRFD	76.314	34	1.49E-4

Source: software results

4 CONCLUSION AND FUTURE WORK

In this study, we have introduced new preconditioned iterative methods based on fractional rotated finite difference method for solving 2D time-fractional diffusion equation. From observation of all experimental results, it can be conclude that the proposed P_2FRFD method requires less time and iterations number when compared to FRFD and P_1FRFD methods with same levels of precision. Therefore, the proposed scheme P_2FRFD may be a good alternative to solve this type of equations and many other numerical problems. Numerical results strongly suggest that the efficiency of the proposed preconditioning methods. The convergence analysis of the present iterative method regarding solutions for 2D time-fractional diffusion equation is currently under study. Furthermore, the idea of this proposed method can be extended to group iterative solver which will be reported separately in the future.

Funding: The author gratefully acknowledges Qassim University, represented by the Deanship of Scientific Research, on the material support for this research under the number (1123-cos-2016-12-s) during the academic year 1438 AH/2017AD).
Competing Interests: The author declares that there are no conflicts of interests regarding the publication of this paper.

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