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THE MARSHALL-OLKIN EXTENDED WEIBULL-EXPONENTIAL DISTRIBUTION: PROPERTIES AND APPLICATIONS

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ABSTRACT

In this paper, a new probability distribution is introduced following the work of Marshall and Olkin [1]. Sub models of the proposed distribution are also important models used in the literature. Expressions for some of its properties such as limiting behavior, quantile function, moments, moment generating function, order statistics, entropy, and reliability functions are derived. The method of maximum likelihood is used in the estimation of the model parameters. The graphs of the hazard rate function plotted for some values of the parameters show that the distribution can be used to model data which exhibits decreasing, increasing or bathtub hazard rate behavior. Series expression of the probability density function was also obtained which enables the expression of some properties of the new distribution in terms of the properties of the base distribution. The distribution is fitted to two real life datasets to show its flexibility and usefulness. Its goodness-of-fit indices indicate better fit to the datasets than the three other distributions compared with it.

Contribution/ Originality: This study originates a new probability distribution named Marshall-Olkin Extended Weibull-Exponential distribution (MOEWED) which is a four-parameter continuous univariate probability distribution capable of modelling data sets of diverse shapes of distribution including approximately symmetric, left-skewed, right-skewed, J-shape, reversed J-shape and unimodal shapes.

1. INTRODUCTION

In a bid to describe and explain random variables, many statistical models have been conceived and formulated in probability theory. However, many of these well-known and classical distributions have been outperformed by more recently proposed distributions in some real life data sets. Again, the actual distributions of various data sets differ in their shapes, nature of hazard rate and many other characteristics defining the distributions. It is therefore very crucial in data analysis to work with an assumed distribution that is as close as possible to the actual unknown distribution of the data set of interest. Consequently, there has been an active interest among researchers to develop new models that yield more satisfactory fit to data sets of interest.

Lately, new methods of generating univariate continuous distributions are mostly based on the idea of introducing additional parameters to an existing distribution or combining two existing distributions to generate a new and often more flexible distribution.

Popular among these new methods include the exponentiation method by Mudholkar and Srivastava [2] applied by Nadarajah and Kotz [3]; Gupta and Kundu [4]; Nadarajah [5] and Flaih, et al. [6] etc. Other methods include the method of adding a parameter to a distribution introduced by Marshall and Olkin [1] the Beta-generated method [7] and the Quadratic rank transmutation map method [8]. Whereas the methods developed by Marshall and Olkin [1] and Shaw and Buckley [8] introduce additional parameter to a baseline distribution, the method introduced by Eugene, et al. [7] combines two existing distributions to generate a new distribution. Extensions of the Beta-generated method have been proposed by Jones [9] and Alzaghal, et al. [10]. Afify, et al. [11]; Khan, et al. [12] and Odom, et al. [13] among many others, have applied the quadratic rank transmutation map method. Using the method introduced by Marshall and Olkin [1] new distributions have also been proposed and studied by Ghitany, et al. [14]; Gui [15]; Krishna, et al. [16]; Al-Saiari, et al. [17]; Benkhelifa [18] and Mansoor, et al. [19]. Also, Santos-Neto, et al. [20] proposed the Marshall-Olkin Extended Weibull family of distributions and studied various properties of the new family of distributions.

Given the fact that the behavior of observed data usually exhibits some sort of departure from that of the theoretical distributions used to model them, this research aims at increasing the flexibility of the Weibull-Exponential distribution (proposed by Oguntunde, et al. [21])) using the method introduced by Marshall and Olkin [1]. This is in response to the well-recognized need to approximate the empirical distribution of available data sets as closely as possible. Oguntunde, et al. [21] in a bid to increase the flexibility of the exponential distribution, used the Weibull generalized family of distributions introduced by Bourguignon, et al. [22] to generate the Weibull-Exponential distribution.

The rest of the paper is structured as follows: the proposed distribution is introduced in section 2.0, section 3.0 presents some mathematical properties of the distribution, section 4.0 provides the estimation and application to real life data while the summary and conclusion is presented in section 5.0.

2. THE MARSHALL-OLKIN EXTENDED WEIBULL-EXPONENTIAL DISTRIBUTION

According to Marshall and Olkin [1] given $G(x)$, $g(x) = dG(x)/dx$ and $\bar{G}(x) = 1 - G(x)$ as the baseline cumulative distribution function (c.d.f.), probability density function (p.d.f.) and survival function, respectively, of a continuous random variable X, the survival function of the Marshall-Olkin extended distribution is given by 1

$$\bar{F}(x) = \frac{\theta \bar{G}(x)}{1-(1-\theta)\bar{G}(x)}, \qquad -\infty < x < \infty, 0 < \theta < \infty \tag{1}$$

This paper uses the above method of Marshall and Olkin [1] to extend the Weibull-Exponential distribution proposed by Oguntunde, et al. [21].

Oguntunde, et al. [21] proposed the Weibull-Exponential distribution (WED) with cumulative distribution function, probability density function and survival functions given respectively as 2-4:

$$G(x) = 1 - \exp\left[-\alpha\left(e^{\lambda x} - 1\right)^{\beta}\right], \qquad x > 0, \alpha, \beta, \lambda > 0 \tag{2}$$

$$g(x) = \alpha\beta\lambda\left(1 - e^{-\lambda x}\right)^{\beta-1} e^{\lambda\beta x} \exp\left[-\alpha\left(e^{\lambda x} - 1\right)\beta\right] \tag{3}$$

$$s(x) = \exp\left[-\alpha\left(e^{\lambda x} - 1\right)^{\beta}\right] \tag{4}$$

Applying 4 in 1, the survival function of the Marshall-Olkin Extended Weibull-Exponential Distribution (MOEWED) is obtained as 5

$$\bar{F}(x)=\frac{\theta\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]} \tag{5}$$

Using the well-known relation, $F(x)=1-\bar{F}(x)$, where $F(x)$ is the c.d.f. of the random variable X, the c.d.f. of the MOEWED is given by 6:

$$F(x;\xi)=\frac{1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}, \qquad x>0, \xi=(\alpha,\beta,\lambda,\theta) \text{ and } \alpha,\beta,\lambda,\theta>0 \tag{6}$$

Consequently, the p.d.f. of the MOEWED is obtained as:

$$f(x;\xi)=\frac{dF(x)}{dx}=\frac{\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{2}}, \qquad x>0, \alpha,\beta,\lambda,\theta>0 \tag{7}$$

where λ is a scale parameter and α, β and θ are shape parameters.

Figure 1 shows the graph of the p.d.f. of Marshall-Olkin Extended Weibull-Exponential distribution. It can be seen from Figure 1 that the Marshall-Olkin Extended Weibull-Exponential density is both right and left skewed, can be unimodal or assume either J shape or reversed-J shape.

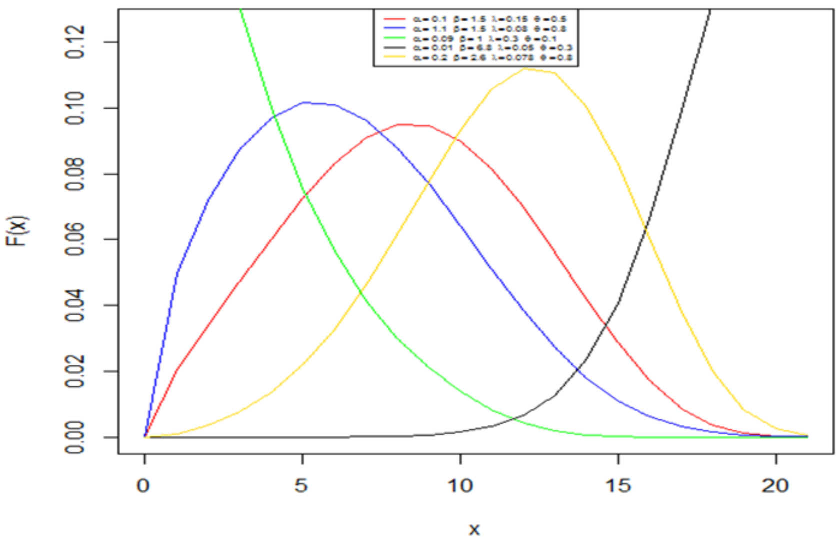

Figure-1. The p.d.f. of the marshall-olkin extended weibull-exponential distribution.

The following sub models are contained in the Marshal-Olkin Extended Weibull-Exponential distribution:

(1) If $\theta=1$, then the MOEWED becomes the Weibull-Exponential distribution.

(2) If $\theta=1$, $\beta=1$ and $\alpha=\theta/\lambda$, then the MOEWED reduces to the Gompertz distribution.

2.1. Limiting Behavior of the MOEWED

The limits of the MOEWED, as $x\to 0$ and as $x\to\infty$ respectively, are investigated.

$$\lim_{x\to 0} = \lim_{x\to 0}\left[\frac{\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{2}}\right]$$

$$= \frac{\alpha\beta\lambda\theta*(0)*(1)*(1)}{[1-(1-\theta)*(1)]^{2}} = 0$$

$$\lim_{x\to\infty} f(x;\xi) = \lim_{x\to\infty}\left[\frac{\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{2}}\right]$$

$$= \frac{\alpha\beta\lambda\theta*(\infty)*(\infty)*(0)}{[1-(1-\theta)*(0)]^{2}} = 0$$

Therefore, with $x>0$, $0\le f(x;\xi)\le 1$ and $\lim_{x\to 0} f(x;\xi) = \lim_{x\to\infty} f(x;\xi) = 0$, the Marshal-Olkin Extended Weibull-Exponential distribution has at least one mode.

2.2. Series Expression of the Probability Density Function of the MOEWED

Theorem 2.1

Let $X \sim MOEWED(\alpha,\beta,\lambda,\theta)$ with p.d.f. as in 7. Therefore the p.d.f. of X can be expressed as an infinite linear combination of the Weibull-Exponential densities as follows 8-10:

$$f(x;\xi) = \sum_{j=0}^{\infty} v_j(\theta)\, g(x;(j+1)\alpha,\beta,\lambda) \tag{8}$$

where $g(x;(j+1)\alpha,\beta,\lambda)$ denotes the Weibull-Exponential density function with parameters $(j+1)\alpha,\beta$ and λ and

$$v_j(\theta) = \begin{cases} \eta_j(\theta) & for \quad 0<\theta<1 \\ q_j(\theta) & for \quad \theta>1 \\ 0 & for \quad \theta=1 \end{cases} \tag{9}$$

where $\eta_j(\theta) = \theta\bar{\theta}^{\,j}$, $\bar{\theta} = 1-\theta$ and

$$q_j(\theta) = \frac{(-1)^j}{\theta(j+1)!}\sum_{k=j}^{\infty}\frac{(k+1)!\left(1-\frac{1}{\theta}\right)^k}{(k-j)!}.$$

Proof

Recall the generalized binomial expansion

$$(1-z)^{-a} = \sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k, \qquad \text{for } |z|<1 \tag{10}$$

where

$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\cdots(a+k-1)$ is the ascending factorial.

For $0<\theta<1$ and $\bar{\theta}=1-\theta$ and applying 10 to the inverse of the denominator of (7), we obtain 11-12:

$$\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{-2} = \sum_{j=0}^{\infty}(j+1)\bar{\theta}^{j}\exp\left[-j\alpha\left(e^{\lambda x}-1\right)^{\beta}\right] \quad (11)$$

Therefore

$$f(x;\xi)=\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\sum_{j=0}^{\infty}(j+1)\bar{\theta}^{j}\exp\left[-j\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]$$

$$=\sum_{j=0}^{\infty}\theta\bar{\theta}^{j}\left[\alpha\beta\lambda(j+1)\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-(j+1)\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]$$

$$f(x;\xi)=\sum_{j=0}^{\infty}\eta_j(\theta)g(x;(j+1)\alpha,\beta,\lambda) \quad (12)$$

where $g(x;(j+1)\alpha,\beta,\lambda)$ denotes the Weibull-Exponential density function with parameters $(j+1)\alpha,\beta$ and λ and $\eta_j(\theta)=\theta\bar{\theta}^{j}$.

For $\theta>1$, consider $1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]$. It can be expressed as

$$1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right] = \theta\left[1-(1-1/\theta)\left(1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right)\right]$$

and $f(x;\xi)$ can therefore be expressed as

$$f(x;\xi)=\frac{\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[\theta\left[1-(1-1/\theta)\left(1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right)\right]\right]^{2}}$$

$$=\frac{\alpha\beta\lambda\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\theta\left[1-(1-1/\theta)\left(1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right)\right]^{2}}.$$

It can be verified that for $\theta > 1$ and $\alpha, \beta, \lambda > 0$, $\left|(1-1/\theta)\left(1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right)\right| < 1$. Therefore, applying (10) to $\left[1-(1-1/\theta)\left(1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right)\right]^{-2}$ and simplifying further, we obtain 13:

$$f(x;\xi) = \frac{1}{\theta}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}(-1)^{j}(k+1)(1-1/\theta)^{k}\alpha\beta\lambda\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-(j+1)\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]$$

$$= \sum_{j=0}^{\infty}\sum_{k=j}^{\infty}(-1)^{j}\theta^{-1}\frac{(1-1/\theta)^{k}}{(k-j)!}\frac{(k+1)!}{(j+1)!}\alpha\beta\lambda(j+1)\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-(j+1)\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]$$

$$f(x;\xi) = \sum_{j=0}^{\infty} q_{j}(\theta)g(x;(j+1)\alpha,\beta,\lambda) \tag{13}$$

where $g(x;(j+1)\alpha,\beta,\lambda)$ denotes the Weibull-Exponential density function with parameters $(j+1)\alpha,\beta$ and λ and

$$q_{j}(\theta) = \frac{(-1)^{j}}{\theta(j+1)!}\sum_{k=j}^{\infty}\frac{(k+1)!\left(1-\frac{1}{\theta}\right)^{k}}{(k-j)!}.$$

For brevity, since $\eta_{j}(\theta)$ and $q_{j}(\theta)$ depend only on the parameter θ, we can write 14:

$$v_{j}(\theta) = \begin{cases}\eta_{j}(\theta) & \text{for} \quad 0<\theta<1 \\ q_{j}(\theta) & \text{for} \quad \theta>1 \\ 0 & \text{for} \quad \theta=1\end{cases} \tag{14}$$

Therefore,

$f(x;\xi) = \sum_{j=0}^{\infty} v_{j}(\theta)g(x;(j+1)\alpha,\beta,\lambda)$. End of proof.

Some mathematical properties of the Marshall-Olkin Extended Weibull-Exponential density can therefore be obtained from the properties of the Weibull-Exponential function.

3. SOME MATHEMATICAL PROPERTIES OF THE MOEWED

3.1. Random Sample Generation

Let X be a continuous random variable with density function as in 7. With 7 being positive everywhere in its domain, the distribution function, 6, is monotonically non-decreasing. Consequently, 6 has an inverse. For $F(x;\xi) = p$, we can find x in terms of p where p is uniformly distributed on $[0,1]$. Therefore,

$$F(x;\xi) = 1 - \frac{\theta\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]} = p$$

$$\Rightarrow x = F^{-1}(p) = Q(p) = \ln\left[1+\left[\ln\left[\frac{1-(1-\theta)p}{1-p}\right]^{1/\alpha}\right]^{1/\beta}\right]^{1/\lambda}. \tag{15}$$

where X is a random sample from MOEWED, $p \sim U(0,1)$ and $F^{-1}(p) = Q(p)$ denotes the inverse distribution function or quantile function of MOEWED 15. Hence, if we can generate p, uniformly distributed on $[0,1]$, then we can simulate the random variable, X, with distribution as in 6.

3.2. Moments and Moment Generating Function

The r^{th} moment about origin of a random variable, X, following the Marshall-Olkin Extended Weibull-Exponential distribution is given by 16:

$$E(X^r) = \int_0^\infty x^r f(x;\xi)dx$$

$$= \int_0^\infty x^r \sum_{j=0}^\infty v_j g(x;(j+1)\alpha, \beta, \lambda)dx \qquad \text{(Applying (8))}$$

$$= \sum_{j=0}^\infty v_j \int_0^\infty x^r g(x;(j+1)\alpha, \beta, \lambda)dx$$

$$E(X^r) = \sum_{j=0}^\infty v_j E(Y_j^r) \tag{16}$$

where $Y_j \sim WE((j+1)\alpha, \beta, \lambda)$ implies that Y_j is a random variable having the Weibull-Exponential density function $g(y_j;(j+1)\alpha, \beta, \lambda)$.

Similarly, the moment generating function, $M_X(t)$, of a random variable, X, following the Marshall-Olkin Extended Weibull-Exponential distribution is given by 17:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x;\xi)dx$$

$$= \int_0^\infty e^{tx} \sum_{j=0}^\infty v_j g(x;(j+1)\alpha, \beta, \lambda)dx$$

$$= \sum_{j=0}^\infty v_j \int_0^\infty e^{tx} g(x;(j+1)\alpha, \beta, \lambda)dx$$

$$M_X(t) = \sum_0^\infty v_j M_{Y_j}(t) \tag{17}$$

where $M_{Y_j}(t)$ is the moment generating function of the random variable, Y_j, having the Weibull-Exponential density function $g(y_j;(j+1)\alpha,\beta,\lambda)$.

3.3. Hazard Rate Function

The hazard rate function of the MOEWED is given by 18:

$$h(x;\xi)=\frac{f(x;\xi)}{1-F(x;\xi)}$$

$$=\frac{\alpha\beta\lambda\theta\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{2}}*\frac{1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\theta\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}$$

$$h(x;\xi)=\frac{\alpha\beta\lambda\left(e^{\lambda x}-1\right)^{\beta-1}e^{\lambda x}}{1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]} \tag{18}$$

Figure 2 shows the graphs of the hazard rate function of the MOEWED for selected values of the parameters. The graphs, having J-shape, reversed-J shape and bathtub shape, shows that the MOEWED can be used to model data that exhibits increasing, decreasing or bathtub hazard rate behaviors. It is J shaped for values of $\beta>1$ and reversed-J shaped or bathtub shaped for values of $\beta<1$.

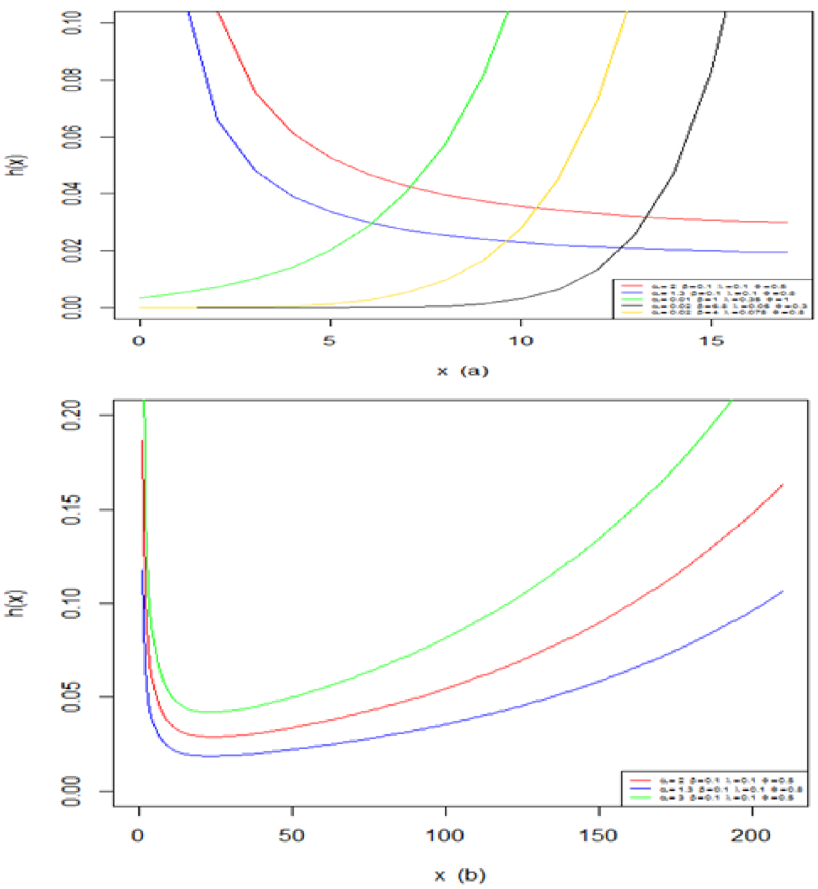

Figure-2. Graphs of the hazard rate function of the MOEWED.

3.4. Entropy

This section considers Rényi entropy which is widely applied in the literature.

Theorem 3.1

The Rényi entropy of a random variable, X, following the Marshal-Olkin Extended Weibull-Exponential distribution is given by 19-20:

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\varphi_j\int_0^{\infty} r(x;\beta,\lambda)^{\gamma}\exp\{-(j+1)\alpha R(x;\beta,\lambda)\}dx\right] \qquad \gamma>0, \gamma\neq 1 \qquad (19)$$

where $R(x;\beta,\lambda)=\left(e^{\lambda x}-1\right)^{\beta}$, $r(x;\beta,\lambda)=R'(x;\beta,\lambda)$ and

$$\varphi_j(\alpha,\theta)=\begin{cases}\tau_j(\alpha,\theta)=\sum_{j=0}^{\infty}\dfrac{\alpha^{\gamma}\theta^{\gamma}\bar{\theta}^{j}(2\gamma)_j}{j!}, & for \quad 0<\theta<1\\[2ex] \omega_j(\alpha,\theta)=\dfrac{(-1)^j\alpha^{\gamma}}{\theta^{\gamma}}\sum_{k=j}^{\infty}\dbinom{k}{j}\dfrac{(2\gamma)_k}{k!}(1-1/\theta)^k, & for \quad \theta>1\\[2ex] 0 & for \quad \theta=1\end{cases} \qquad (20)$$

Proof

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\int_0^{\infty}(f(x;\xi))^{\gamma}dx\right] \qquad \gamma>0, \gamma\neq 1 \qquad (21)$$

$$=\frac{1}{1-\gamma}\ln\left[\int_0^{\infty}\frac{(\alpha\beta\lambda\theta)^{\gamma}\left(e^{\lambda x}-1\right)^{\gamma(\beta-1)}e^{\gamma\lambda x}\exp\left[-\gamma\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{2\gamma}}\right]$$

For $0<\theta<1$ and $\bar{\theta}=1-\theta$ 22-23

$$\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{-2\gamma}=\sum_{j=0}^{\infty}\frac{(2\gamma)_j}{j!}\bar{\theta}^{j}\exp\left[-j\alpha\left(e^{\lambda x}-1\right)^{\beta}\right] \qquad \text{(Applying (10))}$$

Therefore,

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\int_0^{\infty}(\alpha\beta\lambda\theta)^{\gamma}\left(e^{\lambda x}-1\right)^{\gamma(\beta-1)}e^{\gamma\lambda x}\exp\left[-\gamma\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\sum_{j=0}^{\infty}\frac{(2\gamma)_j}{j!}\bar{\theta}^{j}\exp\left[-j\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]dx\right]$$

$$=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\frac{(2\gamma)_j}{j!}\alpha^{\gamma}\theta^{\gamma}\bar{\theta}^{j}\int_0^{\infty}(\beta\lambda)^{\gamma}\left(e^{\lambda x}-1\right)^{\gamma(\beta-1)}e^{\gamma\lambda x}\exp\left[-(j+\gamma)\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]dx\right]$$

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\tau_j(\alpha,\theta)\int_0^{\infty}r(x;\beta,\lambda)^{\gamma}\exp\{-(j+\gamma)\alpha R(x;\beta,\lambda)\}dx\right] \qquad (22)$$

where $R(x;\beta,\lambda)=\left(e^{\lambda x}-1\right)^{\beta}$, $r(x;\beta,\lambda)=R'(x;\beta,\lambda)$ and

$$\tau_j(\alpha,\theta)=\sum_{j=0}^{\infty}\frac{\alpha^{\gamma}\theta^{\gamma}\bar{\theta}^{j}(2\gamma)_j}{j!} \tag{23}$$

For $\theta>1$, following the pattern in subsection 2.2, $\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{-2\gamma}$ can be rewritten as 24

$$\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right]^{-2\gamma} = \theta^{-2\gamma}\left[1-(1-1/\theta)\left\{1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right\}\right]^{-2\gamma}$$

$$=\theta^{-2\gamma}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}(-1)^{j}\frac{(2\gamma)_k}{k!}(1-1/\theta)^{k}\exp\left[-j\alpha\left(e^{\lambda x}-\right)^{\beta}\right]$$

Therefore,

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\int_0^{\infty}(\alpha\beta\lambda\theta)^{\gamma}\left(e^{\lambda x}-\right)^{\gamma(\beta-1)}e^{\gamma\lambda x}\exp\left[-\gamma\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]\right.$$

$$\left.*\,\theta^{-2\gamma}\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}(-1)^{j}\frac{(2\gamma)_k}{k!}(1-1/\theta)^{k}\exp\left[-j\alpha\left(e^{\lambda x}-\right)^{\beta}\right]dx\right]$$

$$=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\binom{k}{j}(-1)^{j}\frac{(2\gamma)_k}{k!}(1-1/\theta)^{k}\theta^{-\gamma}\alpha^{\gamma}\int_0^{\infty}(\beta\lambda)^{\gamma}\left(e^{\lambda x}-\right)^{\gamma(\beta-1)}e^{\gamma\lambda x}\exp\left[-(j+\gamma)\alpha\left(e^{\lambda x}-\right)^{\beta}\right]dx\right]$$

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\omega_j(\alpha,\theta)\int_0^{\infty}r(x;\beta,\lambda)^{\gamma}\exp\{-(j+\gamma)\alpha R(x;\beta,\lambda)\}dx\right] \tag{24}$$

where $R(x;\beta,\lambda)=\left(e^{\lambda x}-1\right)^{\beta}$, $r(x;\beta,\lambda)=R'(x;\beta,\lambda)$ and

$$\omega_j(\alpha,\theta)=\frac{(-1)^{j}\alpha^{\gamma}}{\theta^{\gamma}}\sum_{k=j}^{\infty}\binom{k}{j}\frac{(2\gamma)_k}{k!}(1-1/\theta)^{k}.$$

Again, for brevity, since $\tau_j(\alpha,\theta)$ and $\omega_j(\alpha,\theta)$ depend only on the parameters α and θ, we can write:

$$\varphi_j(\alpha,\theta)=\begin{cases}\tau_j(\alpha,\theta)=\sum_{j=0}^{\infty}\frac{\alpha^{\gamma}\theta^{\gamma}\bar{\theta}^{j}(2\gamma)_j}{j!}, & for \quad 0<\theta<1\\ \omega_j(\alpha,\theta)=\frac{(-1)^{j}\alpha^{\gamma}}{\theta^{\gamma}}\sum_{k=j}^{\infty}\binom{k}{j}\frac{(2\gamma)_k}{k!}(1-1/\theta)^{k}, & for \quad \theta>1\\ 0 & for \quad \theta=1\end{cases}$$

Therefore,

$$T_R(\gamma)=\frac{1}{1-\gamma}\ln\left[\sum_{j=0}^{\infty}\varphi_j\int_0^{\infty}r(x;\beta,\lambda)^{\gamma}\exp\{-(j+1)\alpha R(x;\beta,\lambda)\}dx\right] \qquad \gamma>0,\gamma\neq 1$$

where $\varphi_j(\alpha,\theta)$, $r(x;\beta,\lambda)$ and $R(x;\beta,\lambda)$ are as earlier defined. End of proof.

3.5. Order Statistics

Let $Y_1 \le Y_2 \le Y_3 \le \cdots \le Y_n$ be the order statistics corresponding to the random sample $X_1, X_2, X_3, \cdots, X_n$ from the Marshal-Olkin Extended Weibull-Exponential distribution. Then the p.d.f. of the r^{th} order statistics, Y_r, is given by 26:

$$f_{Y_r}(y) = \frac{n!}{(r-1)!(n-r)!}\left[F(y;\xi)\right]^{r-1}\left[1-F(y;\xi)\right]^{n-r} f(y;\xi) \tag{25}$$

Substituting 6 and 7 in 25, we obtain:

$$f_{Y_r}(y)=\frac{n!\alpha\beta\lambda\theta e^{\lambda x}}{(r-1)!(n-r)!}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\left[\frac{1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}\right]^{r+j-1}\left[\frac{\left(e^{\lambda x}-1\right)^{\beta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}\right] \tag{26}$$

Also, the c.d.f. of the r^{th} order statistics is given by 28:

$$F_{Y_r}(y) = \sum_{i=r}^{n}\binom{n}{i}\left[F(y;\xi)\right]^{i}\left[1-F(y;\xi)\right]^{n-i} \tag{27}$$

Substituting 6 in (27) above, we have:

$$F_{Y_r}(y)=\sum_{i=r}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}(-1)^j\left[\frac{1-\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}{1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x}-1\right)^{\beta}\right]}\right]^{i+j}. \tag{28}$$

4. ESTIMATION AND APPLICATION TO REAL DATA

4.1. Maximum Likelihood Estimation

Let $X_1, X_2, X_3, \cdots, X_n$ be a random sample of size n from the Marshall-Olkin Extended Weibull-Exponential distribution having p.d.f. as in Equation 7. Then the likelihood function, L, of the MOEWED is given by 29:

$$L=\frac{(\alpha\beta\lambda\theta)^n\prod_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta-1}e^{\lambda\sum_{i=1}^{n}x_i}\exp\left[-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\right]}{\prod_{i=1}^{n}\left[1-\bar{\theta}\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right]^2} \tag{29}$$

and the log-likelihood function, $\ln L$, is 30

$$\ln L = n\ln\alpha + n\ln\beta + n\ln\lambda + n\ln\theta + (\beta-1)\sum_{i=1}^{n}\ln\left(e^{\lambda x_i}-1\right)+\lambda\sum_{i=1}^{n}x_i-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}$$

$$-2\sum_{i=1}^{n}\ln\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right] \tag{30}$$

The maximum likelihood estimates (MLE), $\hat{\xi}$ of ξ are therefore obtained as the roots of the following nonlinear equations which are partial derivatives of 30, 31-34 with respect to each of the four parameters

$$\frac{\partial \ln L}{\partial \alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}+2\sum_{i=1}^{n}\frac{(1-\theta)\left(e^{\lambda x_i}-1\right)^{\beta}\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right]} \tag{31}$$

$$\frac{\partial \ln L}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)-\alpha\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)^{\beta}\ln\left(e^{\lambda x_i}-1\right)+2\sum_{i=1}^{n}\frac{\alpha(1-\theta)\left(e^{\lambda x_i}-1\right)^{\beta}\ln\left(e^{\lambda x_i}-1\right)}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right]} \tag{32}$$

$$\frac{\partial \ln L}{\partial \lambda}=\frac{n}{\lambda}+(\beta-1)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}}{\left(e^{\lambda x_i}-1\right)}+\sum_{i=1}^{n}x_i-\alpha\beta\sum_{i=1}^{n}x_i e^{\lambda x_i}\left(e^{\lambda x_i}-1\right)^{\beta-1}$$

$$+2\alpha\beta(1-\theta)\sum_{i=1}^{n}\frac{x_i e^{\lambda x_i}\left(e^{\lambda x_i}-1\right)^{\beta-1}\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right]} \tag{33}$$

$$\frac{\partial \ln L}{\partial \theta}=\frac{n}{\theta}-2\sum_{i=1}^{n}\frac{\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]}{\left[1-(1-\theta)\exp\left[-\alpha\left(e^{\lambda x_i}-1\right)^{\beta}\right]\right]}. \tag{34}$$

These systems of non-linear equations are however, not in closed-form. Therefore, the roots can be obtained numerically using an iterative numerical method such as the Newton-Raphson method Implemented in the maxLik function in R.

4.2. Application to Real Data

In this section, the usefulness of the MOEWED is demonstrated. The goodness-of-fit of the MOEWED is compared with that of the Weibull-Exponential distribution (WED) Oguntunde, et al. [21] Exponentiated Weibull-Exponential distribution (EWED) Alzaatreh, et al. [23] and Gumbel-Weibull distribution (GWD) Al-Aqtash, et al. [24] using two real life data sets. The method of maximum likelihood is used to estimate the parameters of the models and the widely used Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Log-likelihood are used as goodness-of-fit indices. The maximum likelihood estimates, standard errors of the estimates and the goodness-of-fit statistics of the WED, EWED and the GWD are respectively obtained from Oguntunde, et al. [21]; Alzaatreh, et al. [23] and Al-Aqtash, et al. [24].

Data set 1: The first data set, which has been used by Smith and Naylor [25], Bourguignon, et al. [22] and Oguntunde, et al. [21] was originally obtained by workers at the UK National Physical Laboratory. It consists of 63 data points on the strengths of 1.5cm glass fibres. The data is presented in Table 1.

Table-1. Strength of 1.5cm glass fibre data

0.55	0.74	0.77	0.81	0.84	0.93	1.04	1.11	1.13	1.24	1.25	1.27
1.28	1.29	1.3	1.36	1.39	1.42	1.48	1.48	1.49	1.49	1.50	1.50
1.51	1.52	1.53	1.54	1.55	1.55	1.58	1.59	1.60	1.61	1.61	1.61
1.61	1.62	1.62	1.63	1.64	1.66	1.66	1.66	1.67	1.68	1.68	1.69
1.70	1.70	1.73	1.76	1.76	1.77	1.78	1.81	1.82	1.84	1.84	1.89
2.00	2.01	2.24									

Source: Oguntunde, et al. [21].

Table-2. The MLEs, SEs (in parentheses) and the goodness-of-fit indices for the glass fibre data.

Distributions	MOEWED	EWED	WED
Parameter estimates	α = 9.10994 (1.04812)	α = 23.614 (3.954)	α = 0.0175 (0.05746)
	β = 2.43622 (0.31299)	Υ = 7.249 (0.994)	β = 2.87962 (1.94066)
	λ = 0.31454 (0.01877)	c = 0.0033 (0.003)	λ = 1.01779 (1.13950)
	θ = 18.91277 (3.23214)		
Log likelihood	-11.98	−14.33	-14.40
AIC	32.0	34.7	34.8
BIC	40.5	41.1	41.2

Source: Oguntunde, et al. [21]; Alzaatreh, et al. [23]; Al-Aqtash, et al. [24].

It is obvious from Table 2 that the MOEWED has the lowest values for all the three goodness-of-fit indices indicating that it yielded the best fit for the data set.

Data set 2: The second data set consists of 66 data points on the breaking stress of carbon fibres of 50mm length (GPa). It has been previously used by Nichols and Padgett [26], Cordeiro and Lemonte [27], Al-Aqtash, et al. [24] and Oguntunde, et al. [21]. The data is presented in Table 3.

Table-3. Stress of 50mm carbon fibre data.

0.39	0.85	1.08	1.25	1.47	1.57	1.61	1.61	1.69	1.80	1.84
1.87	1.89	2.03	2.03	2.05	2.12	2.35	2.41	2.43	2.48	2.50
2.53	2.55	2.55	2.56	2.59	2.67	2.73	2.74	2.79	2.81	2.82
2.85	2.87	2.88	2.93	2.95	2.96	2.97	3.09	3.11	3.11	3.15
3.15	3.19	3.22	3.22	3.27	3.28	3.31	3.31	3.33	3.39	3.39
3.56	3.60	3.65	3.68	3.70	3.75	4.20	4.38	4.42	4.70	4.90

Source: Oguntunde, et al. [21].

Table-4. The MLEs, SEs (in parentheses) and the goodness-of-fit indices for the carbon fibre data

Distributions	MOEWED	GWD	WED
Parameter estimates	α = 65.79427 (2.12723)	α = 2.4231 (0.5078)	α = 5.25929 (7.54600)
	β = 1.43855 (0.14634)	β = 3.4359 (1.1494)	β = 2.80643 (0.31699)
	λ = 0.04372 (0.00772)	λ = 1.1324 (0.4524)	λ = 0.14236 (0.05404)
	θ = 32.20515 (0.11712)	σ = 5.5673 (2.8064)	
Log likelihood	-84.74	− 84.83	-85.88
AIC	177.5	177.7	177.8
BIC	186.2	186.4	184.3

Source: Oguntunde, et al. [21]; Alzaatreh, et al. [23]; Al-Aqtash, et al. [24].

Table 4 indicates clearly that the MOEWED has lowest values in two out of the three goodness-of-fit indices (lnL and AIC), indicating that it is a strong competitor to the other two distributions, especially the WED.

5. SUMMARY AND CONCLUSION

This work extends the Weibull-Exponential distribution using the method of adding a parameter to a distribution proposed by Marshall and Olkin [1]. Various mathematical properties of the resulting new distribution were studied. Particularly, series expression of the probability density function was derived which makes it possible to obtain some properties of the new distribution in terms of the properties of the base distribution. Maximum likelihood estimation method was used to obtain the parameter estimates. The usefulness of the new distribution was evaluated on the basis of two real life data sets. Its goodness-of-fit indices show its better fit to the data sets than the other distributions compared with it.

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