Journal of Asian Scientific Research
ISSN(e): 2223-1331
ISSN(p): 2226-5724
DOI: 10.55493/5003.v12i4.4646
Vol. 12, No. 4, 188-217.

URL: www.aessweb.com

SPECIAL RELATIVITY IN SIX DIMENSIONS

check for updates

Hilmi Unlu

Istanbul Technical University, Department of Physics Engineering, Faculty of Science and Letters, Maslak /Istanbul, Turkey.
Email: hunlu@itu.edu.tr

Article History
Received: 8 August 2022
Revised: 27 September 2022
Accepted: 10 October 2022
Published: 4 November 2022

Keywords
Doppler shift
Energy
Field transformation
Mass
Maxwell wave equations
Special relativity
Time dilation
Velocity transformation.

ABSTRACT

In the four-dimensional spacetime theory of special relativity, the space coordinate is time contracted along the motion, while perpendicular coordinates are invariant and time varies with position. This leads to a velocity transformation valid at speed of light and used in showing invariance electric and magnetic fields which are invariant along x-axis but change occur along y, and z-axes, contrary to the classical electrodynamics. In this work we introduce a new six-dimensional spacetime theory which allows time (position) change of position (time) in three coordinate axes and still satisfy the Lorentz invariance conditions of metric and Maxwell's wave equations between two frames. We derive a new velocity transformation rule which is valid at any relative speed of massive frames moving with respect to each other. We derived expressions for relativistic mass, energy, Doppler shift, time dilation, length contraction, photon rest mass, and used the conservation of relativistic power to prove that the electric and magnetic fields and consequently, Maxwell wave equations are Lorentz invariant between two massive frames with and without nonzero photon mass in vacuum and materials medium. Calculated photon mass is in excellent agreement with the measured and observed upper bounds of $1.24\text{x}10^{-54}$ kg and $1.75\text{x}\ 10^{-53}$ kg, respectively.

Contribution/ Originality: In this work we extended the four-dimensional spacetime theory to six dimensions by adding two extra time coordinates, which allows spatial time (position) change in position (time) in three coordinate axes and still satisfy the covariance and invariance conditions of the metric and Maxwell's wave equations between two frames under Voigt and Lorentz transformations, respectively. We introduce a new velocity transformation rule which is valid at any relative speed of reference frames moving with respect to each other. We derived expressions for relativistic mass, energy, Doppler shift, time dilation, length contraction, and photon rest mass. For the first time, we found an excellent agreement between the calculated and measured and observed upper bounds of photon mass.

1. INTRODUCTION

Einstein's special theory of relativity [1] removes the difficulties associated with the Galilean transformation of spacetime coordinates in Newtonian mechanics and electromagnetism by using two postulates; (i) The first postulate states that all of the physics laws are the same in all of the inertial frames in which a particle will be at rest or in a state of uniform motion with constant velocity unless there is a net force acting on it. (ii) The second postulate states that the speed of light in a vacuum is the same in all inertial frames, independent of the direction of propagation of electromagnetic waves and of the relative velocity between the light source and observer. In applying Einstein's two postulates, the transformations of the four-dimensional spacetime coordinates and in turn the Cartesian components of velocity between two reference frames are essential for a reliable understanding and precise calculations of the relativistic effects on the physical parameters such as time dilation, length contraction, relativistic and rest masses, momentum, energy dispersion relation of particles, Doppler shift, and invariance of electric and magnetic fields and Maxwell's electromagnetic wave equations.

In the discussion of the abovementioned formulations in the frame of Einstein's special theory of relativity [1] one begins with the idea of thought experiment in which a light source at rest in the $S(x, y, z, t)$ inertial frame with four dimensional spacetime coordinates (x, y, z, t), moving with constant speed c in negative direction, as seen from another four dimensional $S'(x', y', z', t')$ inertial frame with four dimensional spacetime coordinates (x', y', z', t'), is flashed on and off rapidly at $t = t' = 0$. The observers in both inertial frames will see a spherical shell of light radiation expanding outward from the respective origin with constant speed c in all directions. The wave fronts will reach points $P'(x', y', z', t')$ and $P(x, y, z, t)$ in the S' and S inertial frame, respectively, which are described by the following equations [2].

$$s'^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0; \qquad s^2 = x^2 + y^2 + z^2 - c^2t^2 = 0 \tag{1}$$

Where the spacetime coordinates (x', y', z', t') and (x, y, z, t) are often described by so called Lorentz transformation [3] and rarely by Voigt transformations [4] in literature. Lorentz transformation, relates (x', y', z', t') to (x, y, z, t), or vice versa, according to the following equations for forward and inverse transformations, respectively.

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - vx/c^2\right) \tag{2a}$$

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z', \quad t = \gamma\left(t' + vx'/c^2\right) \tag{2b}$$

Where $\gamma = 1/(1 - v^2/c^2)^{1/2}$ is known as Lorentz factor. Equation 2a suggests a time contraction in the line of motion along the horizontal x-axis, while *y*- and *z*- coordinates are invariant ($y' = y$ and $z' = z$). The time decreases by a term that is linear in *x*. Equation 2b is obtained by replacing the prime and unprimed subscripts and *v* with -*v* in Equation 2a for inverse transformation, which suggests a time extension in the line of motion along the horizontal x-axis, while *y*- and *z*- coordinates are invariant ($y' = y$ and $z' = z$). The time increases by a term that is linear in *x*. Equation 2a and 2b keep the following metric and Maxwell's wave equations invariant between two frames [2].

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 \tag{3}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = \left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\varphi' \tag{4}$$

Where $\varphi = \varphi(x, y, z, t)$ and $\varphi' = \varphi'(x', y', z', t')$ are the associated scalar continuous wave functions respectively, and satisfy following conditions at a point in the S and S' inertial frames [5]:

$$\varphi = \varphi'; \qquad \frac{\partial^2\varphi}{\partial x_i} = \frac{\partial^2\varphi'}{\partial x_i}; \qquad \frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi'}{\partial t^2} \tag{5}$$

Where $x_i = x, y, z$. In the frame of Voigt transformation, the spacetime coordinates (x', y', z', t') are written in terms of (x, y, z, t) according to the following set of linear equations [1, 6-8].

$$x' = x - vt, \quad y' = y/\gamma, \quad z' = z/\gamma, \quad t' = \left(t - vx/c^2\right) \tag{6a}$$

$$x = \gamma^2\left(x' + vt'\right), \quad y = \gamma y', \quad z = \gamma z', \quad t = \gamma^2\left(t' + vx'/c^2\right) \tag{6b}$$

Where spacetime coordinates in Equation 6b are obtained by back substitution [6, 7] not by replacing the prime and unprimed subscripts and v with $-v$ in Equation 6a, which is the case in obtaining Equation 2b for inverse Lorentz transformation. Furthermore, Equations 6a and 6b keeps the homogeneous Maxwell's electromagnetic wave equation conformally invariant between the S and S' frames [1].

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi = g_{\mu\nu}\left(\nabla'^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\right)\varphi' \tag{7}$$

Additionally, Voigt transformation also predicts the Doppler effect which is identical to that predicted by special relativity [9]. This suggests that Voigt and Lorentz transformations are closely related, as discussed by Heras from a conceptual point of view [8].

At this junction, it is important to note that multiplying both side of Equations 6a by Lorentz factor γ, one obtains $x'_{iV} = \gamma^{-1}x'_{iL}$ for forward transformation. However, dividing both sides of Equations 6b by γ one obtains $x_{iV} = \gamma x_{iL}$ for inverse transformation. Here x'_{iV} (x'_{iL}) and x_{iV} (x_{iL}) are the spacetime coordinates given by Equations 6a, 2a and 6b, 2b in the frame of Voigt (Lorentz) forward (inverse) transformations, with $x'_i = x', y', z', t'$ and $x_i = x, y, z, t$. That is, the spacetime coordinates in Voigt and Lorentz transformations differ from each other by Lorentz factor γ.

In the frame of the four-dimensional Minkowski-Einstein spacetime, the following metric relation is satisfied for the differential line elements in the S and S' inertial frames [1].

$$dx^2 + dy^2 + dz^2 - c^2dt^2 = g_{\mu\nu}(v)dx'^{\mu}dx'^{\nu} = \frac{1}{1 - v^2/c^2}\left(dx'^2 + dy'^2 + dz'^2 - c^2dt'^2\right) \tag{8}$$

Where $x'^{\mu} = (x', y', z', ct')$ are contravariant 4-vectors and $g^{1/2}_{\mu\nu}(v) = \gamma(v)$ is metric tensor [1].

$$g_{\mu\nu}(v) = \begin{pmatrix} g_{xx} & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 \\ 0 & 0 & g_{zz} & 0 \\ 0 & 0 & 0 & g_{tt} \end{pmatrix} = \frac{1}{1 - v^2/c^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{9}$$

Which suggests a difference between Voigt and Lorentz transformations by Lorentz factor γ.

It is important to note that by using the spacetime coordinates in Equations 2a, 2b and 6a, 6b one finds the following transformation equations for the Cartesian components of the relativistic velocity vector in the S' and S frames, respectively [1-9].

$$u'_x = \frac{dx'}{dt'} = \frac{u_x - v}{\left(1 - u_x v/c^2\right)}, \quad u'_y = \frac{dy'}{dt'} = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)}, \quad u'_z = \frac{dz'}{dt'} = \frac{u_z}{\gamma\left(1 - u_x v/c^2\right)} \tag{10a}$$

$$u_x = \frac{dx}{dt} = \frac{u'_x + v}{\left(1 + u'_x v / c^2\right)}, \quad u_y = \frac{dy}{dt} = \frac{u'_y}{\gamma\left(1 + u'_x v / c^2\right)}, \quad u_z = \frac{dz}{dt} = \frac{u'_z}{\gamma\left(1 + u'_x v / c^2\right)} \tag{10b}$$

Since in each expression in Equation 10a and 10b there are two unknowns, one sets $u_x = c$, $u_y = 0$ and $u_z = 0$ in Equation 10a to find $u'_x = c$ for the one-dimensional motion along + x axis in the S' frame.

Likewise, one sets $u'_x = c$, $u'_y = 0$ and $u'_z = 0$ in Equation 10b to find $u_x = c$ for the one-dimensional motion along -x axis in the S frame. This prompts further investigation of the derivation of the transformation equations for the Cartesian components of the relativistic velocity in both frames.

The similarities between Voigt and Lorentz transformations have been subject to critical discussions from conceptual point of view over the years [1, 6-8]. In this work we only focus on the mathematical solutions of Equation 10a and 10b for the Cartesian components of the relativistic velocity in the S' and S frames, respectively, and its consequences in finding analytical expressions for physical quantities such as time dilation, length contraction, relativistic mass, momentum, energy dispersion relation, Doppler shift, and invariance of electric and magnetic fields and in turn the homogeneous Maxwell's electromagnetic wave equations in the special theory of relativity.

In section 2, we first derive a new six-dimensional spacetime coordinates in Voigt-Lorentz transformation by adding two extra time coordinates to the classical four-dimensional spacetime coordinates [10, 11]. In sections 3 we give the details of the derivation of a new six-dimensional relativistic velocity transformation rule. In section 4 we derive expressions for the relativistic time dilation and length contraction. In section 5 we derive analytical expression for the relativistic mass, momentum, energy dispersion relation, respectively. In section 6, we give the details of the six-dimensional formulation of Doppler shift and show that there is strong correlation between Doppler shift and energy dispersion relation, respectively. In section 7, we give a discussion of the Lorentz invariance of electric and magnetic fields and Maxwell wave equations in vacuum and materials medium as consequences of the six-dimensional spacetime coordinates in special theory of relativity.

2. A SIX-DIMENSIONAL VOIGT-LORENTZ TRANSFORMATION

Some time ago, Recami and Mignani [12]; Demers [13]; Mignani and Recami [14]; Cole [15]; Dattoli and Mingani [16]; Pappas [17]; Teli [18]; Guy [19]; Franco and Jorge [20] added two extra time coordinates to the 4-dimensional spacetime coordinates to interpret the imaginary quantities in the superluminal Lorentz transformations. Time is taken as a vector in the Euclidian 3-dimensional space T^3, so that an event can be represented in Euclidian 6-dimensional space $M^6 = R^3 \times (icT^3)$ as $P \equiv (x, y, z, cit_x, cit_y, cit_z)$. Cartesian components of position vector do not have any meaning for tachyons [13] but the magnitude of time vector $t = (t_x^2 + t_x^2 + t_x^2)^{1/2}$ is observable for bradyons Mignani and Recami [14]. Pappas [17]; Teli [18]; Guy [19] and Franco and Jorge [20] used the time vector as $\vec{t} = (t_x, t_y, t_z)$ in Euclidian 3-dimensional time space T^3 so that an event can be represented in a six dimensional Euclidian spacetime $M^6 = R^3 \times (cT^3)$ as

$P \equiv (x, y, z, ct_x, ct_y, ct_z)$, with a set of linear coordinate equations in six-dimensional $S_6' = S_6'(x', y', z', t_x', t_y', t_z')$ and $S_6 = S_6(x, y, z, t_x, t_y, t_z)$ frames [17, 18].

$$x_i' = \gamma_i (x_i - v_i t_i); \qquad t_i' = \gamma_i (t_i - \frac{v_i}{c^2} x_i); \qquad x_i = \gamma_i (x_i' + v_i t_i'); \qquad t_i = \gamma_i (t_i' + \frac{v_i}{c^2} x_i') \tag{11}$$

where $\gamma_i = 1/(1 - v_i^2 / c^2)^{1/2}$ is anisotropic Lorentz factor, $x_i' = x', y', z'$, $t_{x_i}' = t_x', t_y', t_x'$, and $x_i = x, y, z$

$t_{x_i} = t_x, t_y, t_z$. S_6' and S_6 frames are considered as "massive inertial frames" such as a laboratory or observatory in which a free body is observed to retain its motion [17].

In this section we add two time coordinates to four dimensional spacetime [9] to develop a set of six-dimensional linear spacetime coordinates based on the assumption that an event is taking place in a S_6 massive frame and recorded in S_6' massive frame, or vice versa, which are coincident with a common inertial reference frame at $t = t' = 0$. In doing so, we will search for functional relationships defined as:

$x_i' = x_i'(x_i, t_{x_i})$, $t_{x_i}' = t_{x_i}'(t_{x_i}, x_i)$ in the forward and $x_i = x_i(x_i', t_{x_i}')$, $t_{x_i} = t_{x_i}(t_{x_i}', x_i')$ in the inverse transformations, respectively. We will assume that the $S_6^{'}$ massive frame moves relative to the S_6 massive frame with a three-dimensional constant velocity $\vec{v} = (v_x, v_y, v_z)$ accompanied by a three-dimensional time vector $\vec{t} = (t_x, t_y, t_z)$ in six dimensional spherical coordinates. We will assume the magnitude of time vector $t = (t_x^2 + t_x^2 + t_x^2)^{1/2}$ and $t' = (t_x'^2 + t_y'^2 + t_z'^2)^{1/2}$ is measurable in both frames, but (t_x, t_y, t_z) and (t_x', t_y', t_z') are not, which was proposed by Recami and Mignani [12].

In parallel to the Einstein's thought experiment described in the introduction section, we consider a light source at rest at the origin of the S_6 ($S_6^{'}$) massive frame (moving with constant speed *c* in negative (positive) direction, as seen from the $S_6^{'}$ (S_6) massive frame, is flashed on and off rapidly at time $t = t' = 0$. Einstein's second postulate dictates that observers in both frames will see a spherical shell of radiation expanding outward from the respective origin with constant speed *c*. The wave fronts will reach points $P(x, y, z, t_x, t_y, t_z)$ and $P'(x', y', z', t_x', t_x', t_x')$ in the S_6 and $S_6^{'}$ massive frames, respectively which are described by the following metric equations [10, 11].

$$s^2 = x^2 + y^2 + z^2 - c^2 \left(t_x^2 + t_y^2 + t_z^2\right) = 0; \qquad s'^2 = x'^2 + y'^2 + z'^2 - c^2 \left(t_x'^2 + t_y'^2 + t_z'^2\right) = 0 \tag{12}$$

where the spacetime coordinates $(x', y', z', t_x', t_x', t_x')$ and (x, y, z, t_x, t_y, t_z) in the $S_6^{'}$ and S_6 massive frames are, respectively, described by the following set of linear equations [10, 11].

$$x' = x - v_x t_x, \quad y' = y - v_y t_y, \quad z' = z - v_z t_z;$$
$$t'_x = t_x - (v_x / c^2)x, \quad t'_y = t_y - (v_y / c^2)y, \quad t'_z = t_z - (v_z / c^2)z \tag{13a}$$

$$x = x' + v_x t'_x, \quad y = y' + v_y t'_y, \quad z = z' + v_z t'_z;$$
$$t_x = t'_x + (v_x / c^2)x', \quad t_y = t'_y + (v_y / c^2)y', \quad t_z = t'_z + (v_z / c^2)z' \tag{13b}$$

with $v_x = v\cos\phi\sin\theta$, $v_y = v\sin\phi\sin\theta$, $v_z = v\cos\theta$ and $t_x = t\sin\theta\cos\phi$, $t_y = t\sin\theta\sin\phi$,

$t_z = t\cos\theta$ in spherical coordinates. Equation 13a and 13b, respectively, allow the time contraction (extension) of space coordinates and spatial positional decrease (increase) in time coordinates in the $S_6^{'}$ (S_6) massive frame in forward and inverse transformations.

2.1. Six-Dimensional Metric Equations

In this subsection we will use the similarity between Voigt and Lorentz transformations described by Equations 2a, 2b and 6a, 6b to give proof of using coefficients γ_i in the spacetime coordinate equations in Equation 11. In doing so, we start with using Equation 13a and 13b to write the following metric equations between S_6 and $S_6^{'}$ massive frames [10, 11].

$$\begin{aligned} x^2 + y^2 + z^2 - c^2(t_x^2 + t_y^2 + t_z^2) &= g_{\mu\nu}(x'^2 + y'^2 + z'^2) - g_{\mu\nu}c^2(t_x'^2 + t_y'^2 + t_z'^2) \\ &= G_{xx}x^2 + G_{yy}y^2 + G_{zz}(g_{zz} - g_{t_z t_z}\beta_z^2)z^2 - c^2\left(G_{t_x t_x}t_x^2 + G_{t_y t_y}t_y^2 + G_{t_z t_z}t_z^2\right) \end{aligned} \tag{14}$$

where $\beta_x = v_x / c$, $\beta_y = v_y / c$, $\beta_z = v_z / c$. v_x, v_y, and v_z are x, y, and z- components of velocity $\vec{v}$ in spherical polar coordinates. Coefficients $G_{\mu\nu}$ for each term on right hand side of Equation 14 are

$$\begin{aligned} &G_{xx} = \left(g_{xx} - g_{t_x t_x}\beta_x^2\right), \quad G_{yy} = \left(g_{yy} - g_{t_y t_y}\beta_y^2\right), \quad G_{zz} = \left(g_{zz} - g_{t_z t_z}\beta_z^2\right) \\ &G_{t_x t_x} = \left(g_{t_x t_x} - g_{xx}\beta_x^2\right), \quad G_{t_y t_y}\left(g_{t_y t_y} - g_{yy}\beta_y^2\right), \quad G_{t_z t_z} = \left(g_{t_z t_z} - g_{zz}\beta_z^2\right) \end{aligned} \tag{15}$$

Matching both sides of Equation 14 component by component, gives $G_{xx} = 1$, $G_{yy} = 1$, $G_{zz} = 1$ $G_{t_x t_x} = 1$, $G_{t_y t_y} = 1$, and $G_{t_z t_z} = 1$. These results then transform Equation 14 from being covariant to form invariant under Voigt transformation between S_6 and $S_6^{'}$ frames and gives the following six-dimensional Voigt scaling coefficients

$$g_{xx} = g_{t_x t_x} = 1/(1-\beta_x^2), \quad g_{yy} = g_{t_y t_y} = 1/(1-\beta_y^2), \quad g_{zz} = g_{t_z t_z} = 1/(1-\beta_z^2) \tag{16}$$

Using $g_{\mu\nu} = \gamma_{\mu\nu}^2$ in Equation 14 we write the following metric equation between S_6 and $S_6^{'}$ massive frames

$$x^2+y^2+z^2-c^2(t_x^2+t_y^2+t_z^2)=\gamma_{\mu\nu}^2(x'^2+y'^2+z'^2)-\gamma_{\mu\nu}^2c^2(t_x'^2+t_y'^2+t_z'^2)$$
$$=a_{xx}x^2+a_{yy}y^2+a_{zz}z^2-c^2\left(a_{t_xt_x}t_x^2+a_{t_yt_y}t_y^2+a_{t_zt_z}t_z^2\right) \tag{17}$$

where coefficients $a_{\mu\nu}$ are

$$a_{xx}=\left(\gamma_{xx}^2-\gamma_{t_xt_x}^2\frac{v_x^2}{c^2}\right),\quad a_{yy}=\left(\gamma_{yy}^2-\gamma_{t_yt_y}^2\frac{v_y^2}{c^2}\right),\quad a_{zz}=\left(\gamma_{zz}^2-\gamma_{t_zt_z}^2\frac{v_z^2}{c^2}\right)$$
$$a_{t_xt_x}=\left(\gamma_{t_xt_x}^2-\gamma_{xx}^2\frac{v_x^2}{c^2}\right),\quad a_{t_yt_y}=\left(\gamma_{yy}^2-\gamma_{t_yt_y}^2\frac{v_y^2}{c^2}\right),\quad a_{t_zt_z}=\left(\gamma_{t_zt_z}^2-\gamma_{zz}^2\frac{v_z^2}{c^2}\right) \tag{18}$$

Matching both sides of Equation 17 component by component, gives $a_{xx}=1$, $a_{yy}=1$, $a_{zz}=1$ $a_{t_xt_x}=1$, $a_{t_yt_y}=1$, and $a_{t_zt_z}=1$, which transforms Equation 17 from being covariant to invariant under Lorentz transformation between S_6 and $S_6^{'}$ frames and gives following coefficients.

$$\gamma_{xx}=\gamma_{t_xt_x}=1/\sqrt{1-\beta_x^2},\quad \gamma_{yy}=\gamma_{t_yt_y}=1/\sqrt{1-\beta_y^2},\quad \gamma_{zz}=\gamma_{t_zt_z}=1/\sqrt{1-\beta_z^2} \tag{19}$$

As Cartesian components of six-dimensional Lorentz scaling factor. One can then write the following linear expressions for the spacetime coordinates in the S_6' and S_6 frames in Lorentz transformation Ünlü [10], Unlü [11].

$$x'=\gamma_{xx}(x-v_xt_x),\quad y'=\gamma_{yy}(y-v_yt_y),\quad z'=\gamma_{zz}(z-v_zt_z);$$
$$t_x'=\gamma_{t_xt_x}(t_x-\beta_xx/c),\quad t_y'=\gamma_{t_yt_y}(t_y-\beta_yy/c),\quad t_z'=\gamma_{t_zt_z}(t_z-\beta_zz/c) \tag{20a}$$

$$x=\gamma_{xx}(x^{'}+v_xt_x'),\quad y=\gamma_{yy}(y^{'}+v_yt_y'),\quad z=\gamma_{zz}(z^{'}+v_zt_z');$$
$$t_x=\gamma_{t_xt_x}(t_x^{'}+\beta_xx'/c),\quad t_y=\gamma_{t_yt_y}(t_y^{'}+\beta_yy'/c),\quad t_z=\gamma_{t_zt_z}(t_z^{'}+\beta_zz'/c) \tag{20b}$$

Equation 20a allow time contraction (position change) of three space (time) coordinates under forward Lorentz transformation. Equation 20b allow time extension (position increase) of space (time) coordinates under inverse Lorentz transformation. It is gratifying to note that Equation 20a and 20b provide direct and independent proofs of using coefficients in Equation 11.

2.2. Six-Dimensional Maxwell Wave Equations

Applying the chain rules for differential operators of x and t_x in Equation 13a and 13b we can write:

$$\frac{\partial\varphi^{'}}{\partial x}=\frac{\partial\varphi^{'}}{\partial x^{'}}\frac{\partial x^{'}}{\partial x}+\frac{\partial\varphi^{'}}{\partial t_x^{'}}\frac{\partial t_x^{'}}{\partial x};\qquad \frac{\partial\varphi^{'}}{\partial t_x}=\frac{\partial\varphi^{'}}{\partial t_x^{'}}\frac{\partial t_x^{'}}{\partial t_x}+\frac{\partial\varphi^{'}}{\partial x^{'}}\frac{\partial x^{'}}{\partial t_x} \tag{21a}$$

$$\frac{\partial^2\varphi^{'}}{\partial x^2}=\frac{\partial}{\partial x^{'}}\left(\frac{\partial\varphi^{'}}{\partial x}\right)\frac{\partial x^{'}}{\partial x}+\frac{\partial}{\partial t_x^{'}}\left(\frac{\partial\varphi^{'}}{\partial x}\right)\frac{\partial t_x^{'}}{\partial x};\qquad \frac{\partial^2\varphi'}{\partial t_x^2}=\frac{\partial}{\partial x^{'}}\left(\frac{\partial\varphi'}{\partial t_x}\right)\frac{\partial x^{'}}{\partial t_x}+\frac{\partial}{\partial t_x^{'}}\left(\frac{\partial\varphi'}{\partial t_x}\right)\frac{\partial t_x^{'}}{\partial t_x} \tag{21b}$$

With similar expressions for $\partial^2\varphi'/\partial y^2$, $\partial^2\varphi'/\partial z^2$, $\partial^2\varphi'/\partial t_y^2$ and $\partial^2\varphi'/\partial t_z^2$, one then obtains

$$\begin{aligned}\nabla^2\varphi-\frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t_x^2}+\frac{\partial^2\varphi}{\partial t_y^2}+\frac{\partial^2\varphi}{\partial t_z^2}\right)&=g_{\mu\nu}\nabla'^2\varphi'-g_{\mu\nu}\frac{1}{c^2}\left(\frac{\partial^2\varphi'}{\partial t_x'^2}+\frac{\partial^2\varphi'}{\partial t_y'^2}+\frac{\partial^2\varphi'}{\partial t_z'^2}\right)\\&=G_{xx}\frac{\partial^2\varphi'}{\partial x'^2}+G_{yy}\frac{\partial^2\varphi'}{\partial y'^2}+G_{zz}\frac{\partial^2\varphi'}{\partial z'^2}-\frac{1}{c^2}\left(G_{t_xt_x}\frac{\partial^2\varphi'}{\partial t_x'^2}+G_{t_yt_y}\frac{\partial^2\varphi'}{\partial t_y'^2}+G_{t_zt_z}\frac{\partial^2\varphi'}{\partial t_z'^2}\right)\end{aligned} \tag{22}$$

Which are covariant Maxwell wave equations under Voigt transformation between two massive frames, with $G_{\mu\nu}$ are given by Equation 15. Substituting $g_{\mu\nu}=\gamma_{\mu\nu}^2$ into Equation 22 one obtains.

$$\begin{aligned}\nabla^2\varphi-\frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t_x^2}+\frac{\partial^2\varphi}{\partial t_y^2}+\frac{\partial^2\varphi}{\partial t_z^2}\right)&=\gamma_{\mu\iota}^2\nabla'^2\varphi'-\gamma_{\mu\iota}^2\frac{1}{c^2}\left(\frac{\partial^2\varphi'}{\partial t_x'^2}+\frac{\partial^2\varphi'}{\partial t_y'^2}+\frac{\partial^2\varphi'}{\partial t_z'^2}\right)\\&=a_{xx}\frac{\partial^2\varphi'}{\partial x'^2}+a_{yy}\frac{\partial^2\varphi'}{\partial y'^2}+a_{zz}\frac{\partial^2\varphi'}{\partial z'^2}-\frac{1}{c^2}\left(a_{t_xt_x}\frac{\partial^2\varphi'}{\partial t_x'^2}+a_{t_yt_y}\frac{\partial^2\varphi'}{\partial t_y'^2}+a_{t_zt_z}\frac{\partial^2\varphi'}{\partial t_z'^2}\right)\end{aligned} \tag{23}$$

Where $a_{\mu\nu}$ are given by Equation 18. Setting $a_{xx}=1$, $a_{yy}=1$, $a_{zz}=1$ $a_{t_xt_x}=1$, $a_{t_yt_y}=1$, and $a_{t_zt_z}=1$ makes Equation 23 invariant under Lorentz transformation between two massive frames.

3. A SIX-DIMENSIONAL VELOCITY TRANSFORMATION

The transformation of the relativistic velocity components between two frames is essential for a reliable understanding and precise calculations of the relativistic effects on parameters such as time dilation, length contraction, relativistic mass, momentum, energy dispersion relations in massive frames [10, 11]. Equation 13a and 13b or Equation 20a and 20b with $\gamma_{xx}=\gamma_{t_xt_x}$, $\gamma_{yy}=\gamma_{t_yt_y}$ and $\gamma_{zz}=\gamma_{t_zt_z}$, are used to write the following expressions for Cartesian components of velocities $\vec{u}$ and $\vec{u}'$ of an event taking place in the S_6 frame and observed in the S_6' frame

$$u_x'=\frac{dx'}{dt_x'}=\left(u_x-v_x\right)\frac{dt_x}{dt_x'}=\frac{\left(u_x-v_x\right)}{1-u_xv_x/c^2};\qquad u_x=\frac{dx}{dt_x}=\left(u_x'+v_x\right)\frac{dt_x'}{dt_x}=\frac{\left(u_x'+v_x\right)}{1+u_x'v_x/c^2} \tag{24a}$$

$$u_y'=\frac{dy'}{dt_y'}=\left(u_y-v_y\right)\frac{dt_y}{dt_y'}=\frac{u_y-v_y}{1-u_yv_y/c^2};\qquad u_y=\frac{dy}{dt_y}=\left(u_y'+v_y\right)\frac{dt_y'}{dt_y}=\frac{\left(u_y'+v_y\right)}{1+u_y'v_y/c^2} \tag{24b}$$

$$u_z'=\frac{dz'}{dt_z'}=\left(u_z-v_z\right)\frac{dt_z}{dt_z'}=\frac{u_z-v_z}{1-u_zv_z/c^2};\qquad u_z=\frac{dz}{dt_z}=\left(u_z'+v_z\right)\frac{dt_z'}{dt_z}=\frac{\left(u_z'+v_z\right)}{1+u_z'v_z/c^2} \tag{24c}$$

When S_6' moves parallel to x (or y, z) axis of S_6 at the speed of light, Equation 24a, 24b and 24c give $u_x'=-c$ and $u_x=c$ ($u_y'=-c$ and $u_y=c$, $u_z'=-c$ and $u_z=c$).

We can extend Equation 24a, 24b, and 24c to any relative speed between the two massive frames by combining Equation 13a and 13b or Equation 20a and 20b with $\gamma_{xx} = \gamma_{t_x t_x}$, $\gamma_{yy} = \gamma_{t_y t_y}$ and $\gamma_{zz} = \gamma_{t_z t_z}$, for the Cartesian components of velocity vectors $\vec{u}'$ and $\vec{u}$ in the $S_6^{'}$ and S_6 massive frames

$$\begin{aligned} u_x' &= \frac{dx'}{dt_x'} = \left(1-\frac{v_x^2}{c^2}\right)\frac{dx}{dt_x'} - v_x = \left(1-\frac{v_x^2}{c^2}\right)\left(u_x' + v_x\right) - v_x; \\ u_x &= \frac{dx}{dt_x} = \left(1-\frac{v_x^2}{c^2}\right)\frac{dx'}{dt_x} + v_x = \left(1-\frac{v_x^2}{c^2}\right)\left(u_x - v_x\right) + v_x \end{aligned} \tag{25}$$

From which we find $u_x' = -v_x$ and $u_x = v_x$, respectively. Cartesian components of the velocity vector in the S_6 and $S_6^{'}$ massive frames are then written as

$$u_x = v_x = v\cos\phi\sin\theta, \quad u_y = v_y = v\sin\phi\sin\theta, \quad u_z = v_z = v\cos\theta \tag{26a}$$

$$u_x' = -v_x = -v\cos\phi\sin\theta, \quad u_y' = -v_y = -v\sin\phi\sin\theta, \quad u_z' = -v_z = -v\cos\theta; \tag{26b}$$

Equation 26a and 26b suggest that the u_x', u_y', u_z' components of $\vec{u}'$ in the $S_6^{'}$ frame can be determined by using the relative speed of two frames, without requiring one of the unknowns to be known (e.g., $u_x = c$, $u_y = c$ and $u_z = c$) in the S_6 frame. The negative sign in velocity components in $S_6^{'}$ massive frame is consistent with the principle of Einstein's velocity reciprocity relation [9]. We will use Figure 1 to study the influence of the components of velocity vectors onto each other by using the direction cosine [21] for unit vector transformation.

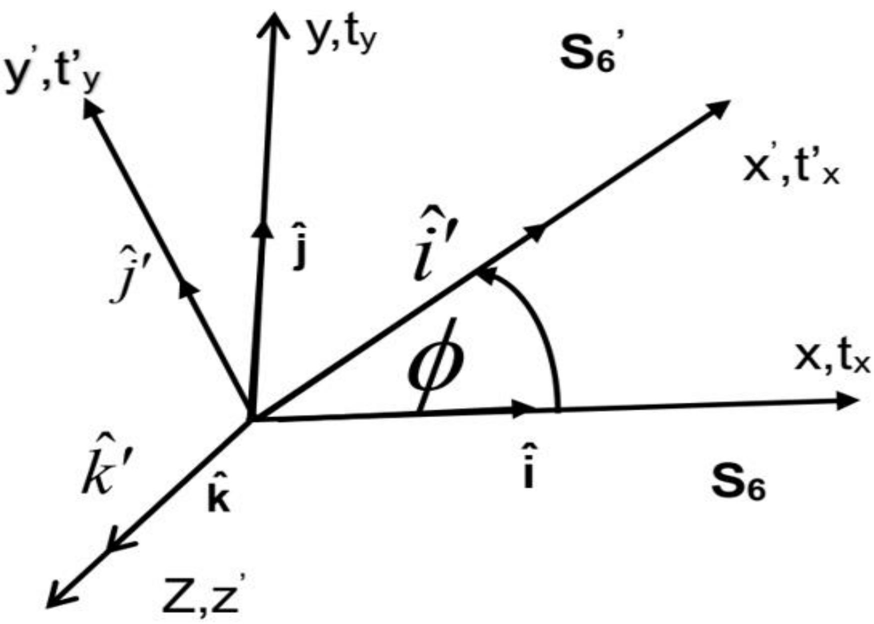


Figure 1. The schematic diagram of unit vector transformation via rotation through angle ϕ in counter clockwise of (x, y) plane into (x', y') plane for $0 \le \phi \le 2\pi$ and $\theta = \pi/2$.

Considering rotation through angle ϕ counterclockwise of (x, y) plane into (x', y') plane with z or z' axis the same, we write $(\hat{i}', \hat{j}', \hat{k}')$ in S_6' in terms of $(\hat{i}, \hat{j}, \hat{k})$ in S_6 frame.

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos(\hat{i}' \cdot \hat{i}) & \cos(\hat{i}' \cdot \hat{j}) & \cos(\hat{i}' \cdot \hat{k}) \\ \cos(\hat{j}' \cdot \hat{i}') & \cos(\hat{j}' \cdot \hat{j}) & \cos(\hat{j}' \cdot \hat{k}) \\ \cos(\hat{k}' \cdot \hat{i}) & \cos(\hat{k}' \cdot \hat{j}) & \cos(\hat{k}' \cdot \hat{k}) \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \tag{27}$$

Replacing ϕ with $-\phi$, rotation through angle ϕ clockwise of (x', y') plane into (x, y) plane with z' or z axis the same then yields $(\hat{i}, \hat{j}, \hat{k})$ in S_6 frame is written in terms of $(\hat{i}', \hat{j}', \hat{k}')$ in S_6' frame:

$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \cos(\hat{i}.\hat{i}') & \cos(\hat{j}.\hat{i}') & \cos(\hat{k}.\hat{i}') \\ \cos(\hat{i}.\hat{j}'\cdot) & \cos(\hat{j}.\hat{j}') & \cos(\hat{k}.\hat{j}') \\ \cos(\hat{i}.\hat{k}') & \cos(\hat{j}.\hat{k}') & \cos(\hat{k}.\hat{k}') \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} \tag{28}$$

Equation 27 and 28 can then be used to write the following expressions for the velocity vectors $\vec{u}$ and $\vec{u}'$ in the S_6 and S_6' massive frames.

$$\vec{u} = u_x\hat{i} + u_y\hat{j} + u_z\hat{k} = \left(u_x' \cos\phi - u_y' \sin\phi\right)\hat{i}' + \left(u_x' \sin\phi + u_y' \cos\phi\right)\hat{j}' + u_z'\hat{k}' \tag{29a}$$

$$\vec{u}' = u_x'\hat{i}' + u_y'\hat{j}' + u_z'\hat{k}' = \left(u_x \cos\phi + u_y \sin\phi\right)\hat{i} + \left(-u_x \sin\phi + u_y \cos\phi\right)\hat{j} + u_z\hat{k} \tag{29b}$$

Figure 2 shows the azimuthal angle variation of Cartesian components of an event taking place in the S_6 massive frame and observed in the S_6' massive frame as a function of azimuthal angle ϕ for $\theta = \pi/6, \pi/4, \pi/3$ and $\theta = \pi/2$, respectively, in system of spherical polar coordinates.

As the light source at the origin of the S_6 massive frame is flashed on and off rapidly, the observers in both frames will see a spherical shell of radiation expanding outward from the origin in all directions.

When S_6' moves relative to S_6 at the speed of light, using Equation 26a and 26b in Equations 29a and 29b gives

$$|\vec{u}| = \left(u_x^2 + u_y^2 + u_z^2\right)^{1/2} = c; \qquad |\vec{u}'| = \left(u_x'^2 + u_y'^2 + u_z'^2\right)^{1/2} = c \tag{30}$$

Which proves that the speed of light in a vacuum is the same in both massive frames, independent of the direction of the wave propagation and of the relative velocity of the two frames, which is in accordance with Einstein's second postulate of special relativity [9].

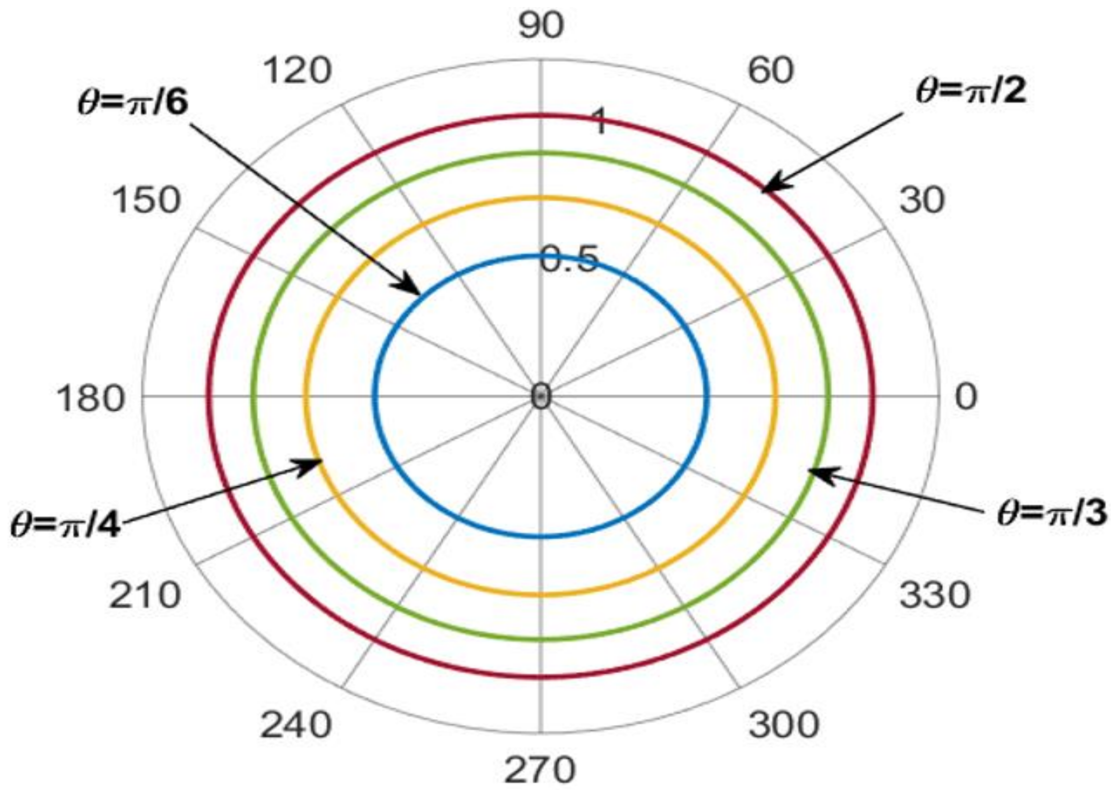


Figure 2. Polar plot of u'_x and u'_y components of $\vec{u}'$ of an event taking place in S_6 frame and observed in the $S_6^{'}$ frame as a function of azimuthal angle ϕ for $\theta = \pi/6, \pi/4, \pi/3$ and $\theta = \pi/2$, respectively, in system of spherical coordinates.

4. TIME DILATION AND LENGTH CONTRACTION

4.1. Time Dilation

In order to explore the physical implications of the six-dimensional spacetime coordinates in special relativity, we write the (6x6) the conformally invariant metric equation for line intervals between the $S_6^{'}$ and S_6 frames under the Voigt transformation:

$$ds^2 = g_{\mu\nu} ds'^2 = g_{\mu\nu} dx'^{\mu} dx'^{\nu} = \begin{pmatrix} g_{xx} & 0 & 0 & 0 & 0 & 0 \\ 0 & g_{yy} & 0 & 0 & 0 & 0 \\ 0 & 0 & g_{zz} & 0 & 0 & 0 \\ 0 & 0 & 0 & g_{t_x t_x} & 0 & 0 \\ 0 & 0 & 0 & 0 & g_{t_y t_y} & 0 \\ 0 & 0 & 0 & 0 & 0 & g_{t_z t_z} \end{pmatrix} dx'^{\mu} dx'^{\nu} \tag{31}$$

where $x'^{\mu} = (x', y', z', t'_x, t'_y, t'_z)$ are the 6-vectors and $g_{\mu\nu}$ is the 6-dimensional analogue of 4-dimensional spacetime metric tenor in Equation 9. Here $g_{xx} = g_{t_x t_x}$, $g_{yy} = g_{t_y t_y}$, $g_{zz} = g_{t_z t_z}$ given in Equation 16. The differential line intervals in the $S_6^{'}$ and S_6 massive frames are written as:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\left(dt_x^2 + dt_y^2 + dt_z^2\right) = -c^2\left[dt_x^2\left(1-\frac{u_x^2}{c^2}\right) - dt_y^2\left(1-\frac{u_y^2}{c^2}\right) - dt_z^2\left(1-\frac{u_z^2}{c^2}\right)\right] \quad (32a)$$

$$g_{\mu\nu}ds'^2 = g_{xx}dx'^2 + g_{yy}dy'^2 + g_{xx}dz'^2 - c'^2\left(g_{t_{xx}}dt_x'^2 + g_{t_{yy}}dt_y'^2 + g_{t_{zz}}dt_z'^2\right)$$
$$= -c'^2\left[g_{xx}dt_x'^2\left(1-\frac{u_x'^2}{c'^2}\right) - g_{yy}dt_y'^2\left(1-\frac{u_y'^2}{c'^2}\right) - g_{zz}d't_z^2\left(1-\frac{u_z'^2}{c'^2}\right)\right] \quad (32b)$$

where $g_{xx} = g_{t_x t_x}$, $g_{yy} = g_{t_y t_y}$, $g_{zz} = g_{t_z t_z}$ given in Equation 16. Matching both sides of Equation 31 gives:

$$\Delta t_x = \left(1-\frac{u_x'^2}{c'^2}\right)^{-1/2}\Delta t_x'; \qquad \Delta t_y = \left(1-\frac{u_y'^2}{c'^2}\right)^{-1/2}\Delta t_y'; \qquad \Delta t_z = \left(1-\frac{u_z'^2}{c'^2}\right)^{-1/2}\Delta t_z' \quad (33)$$

as the Cartesian components of time dilation in the frame of six-dimensional spacetime coordinates.

Figure 3 shows the azimuthal angle variation of normalized Δt_x and Δt_y for $\beta = 0.90$ (Figure 3a) and $\beta = 1$ (Figure 3b), respectively, with polar angles $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

As the azimuthal angle ϕ increases in counterclockwise from +x axis of the S_6 massive frame, $\Delta t_x / \Delta t_x'$ ($\Delta t_y / \Delta t_y'$) decreases (increases) for a given polar angle θ and $\Delta t_x / \Delta t_x' = \Delta t_y / \Delta t_y' = 1/2$ at $\phi = \pi/4$. When the $S_6^{'}$ massive moves along the +y- axis of S_6 frame ($\phi = \pi/2, 3\pi/2$), $\Delta t_x / \Delta t_x'$ and $\Delta t_y / \Delta t_y'$ vary with ϕ for $\theta = \pi/6, \pi/4, \pi/3$ and becomes elliptical for $\theta = \pi/2$.

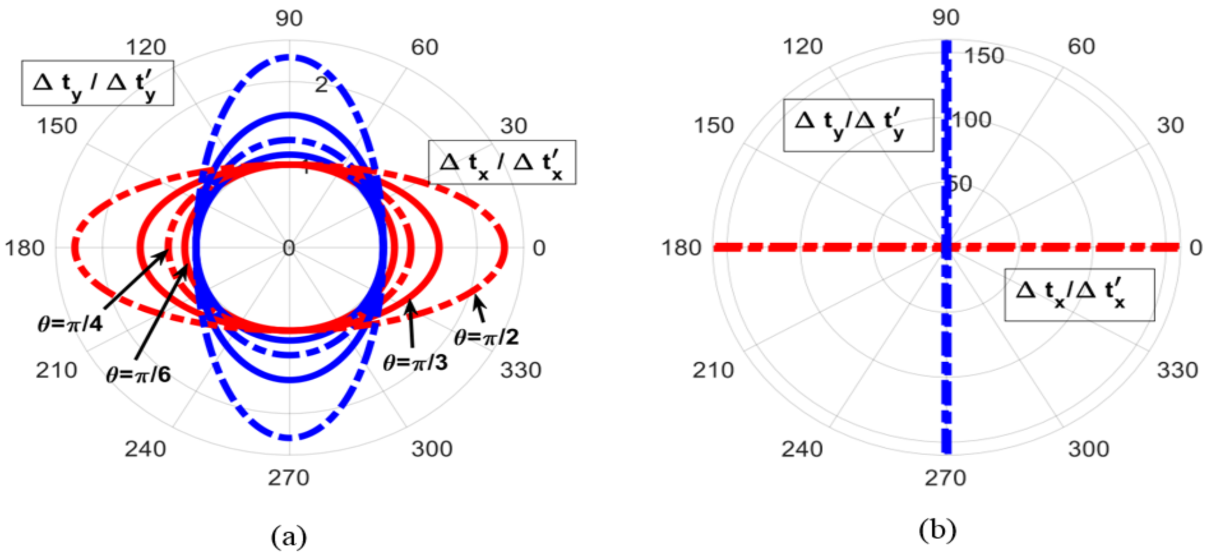


Figure 3. Polar plot of normalized Cartesian components of the time dilation plotted as a function of azimuthal angle ϕ for $\beta = v/c = 0.90$ and $\theta = \pi/6, \pi/4, \pi/3, \pi/2$ (a) and $\beta = v/c = 1$ (b) and $\theta = \pi/6, \pi/4, \pi/3, \pi/2$.

When the $S_6^{'}$ massive frame moves with speed of light ($\beta = 1$) relative to the S_6 massive frame, Figure 3b implies full relativistic regime for $\theta = \pi/6, \pi/4, \pi/3, \pi/2$. When the $S_6^{'}$ massive frame moves relative to the

S_6 massive frame at constant speed smaller than speed of light ($v<c$), Equation 33 suggests time contraction along the direction of motion by a factor γ^{-1} with respect to the corresponding Δt of a clock at rest and measured by the observer in the S_6 massive frame.

As pointed out by Recami and Mignani [12]; Demers [13]; Mignani and Recami [14]; Cole [15]; Dattoli and Mingani [16]; Pappas [17]; Teli [18]; Guy [19]; Franco and Jorge [20] the magnitude of the time vector $t=\left(t_x^2+t_x^2+t_x^2\right)^{1/2}$ and $t'=\left(t_x'^2+t_x'^2+t_x'^2\right)^{1/2}$ can be measured in both frames, but (t_x,t_y,t_z) and (t_x',t_y',t_z') of time can't be. One can use Equation 33 to writes the measurable time dilation as:

$$\Delta t=\left(\Delta t_x^2+\Delta t_y^2+\Delta t_z^2\right)^{1/2}=\left(\frac{1}{1-u_x'^2/c'^2}+\frac{1}{1-u_y'^2/c'^2}+\frac{1}{1-u_z'^2/c'^2}\right)^{1/2}\Delta t' \tag{34}$$

which states that the change $\Delta t'$ is absolute and does not depend on the location of observer.

Figure 4 shows the azimuthal angle variation of the measurable time dilation for varying β with $\theta=\pi/2$ Figure 4a and for varying θ with $\beta=1$ (Figure 4b), respectively. When S_6' massive frame moves with speed of light relative to S_6 massive frame, Equation 33 gives $\Delta t=\sqrt{14/3}\Delta t'$ for $\phi=\pi/4$ and $\theta=\pi/2$. As shown in Figure 4a, when β increases from 0.60 towards unity, time dilation curve becomes more and more parabolic. When polar angle increases from $\theta=\pi/6$ towards $\theta=\pi/2$ for $\beta=1.00$, the slope of time dilation curve becomes steeper.

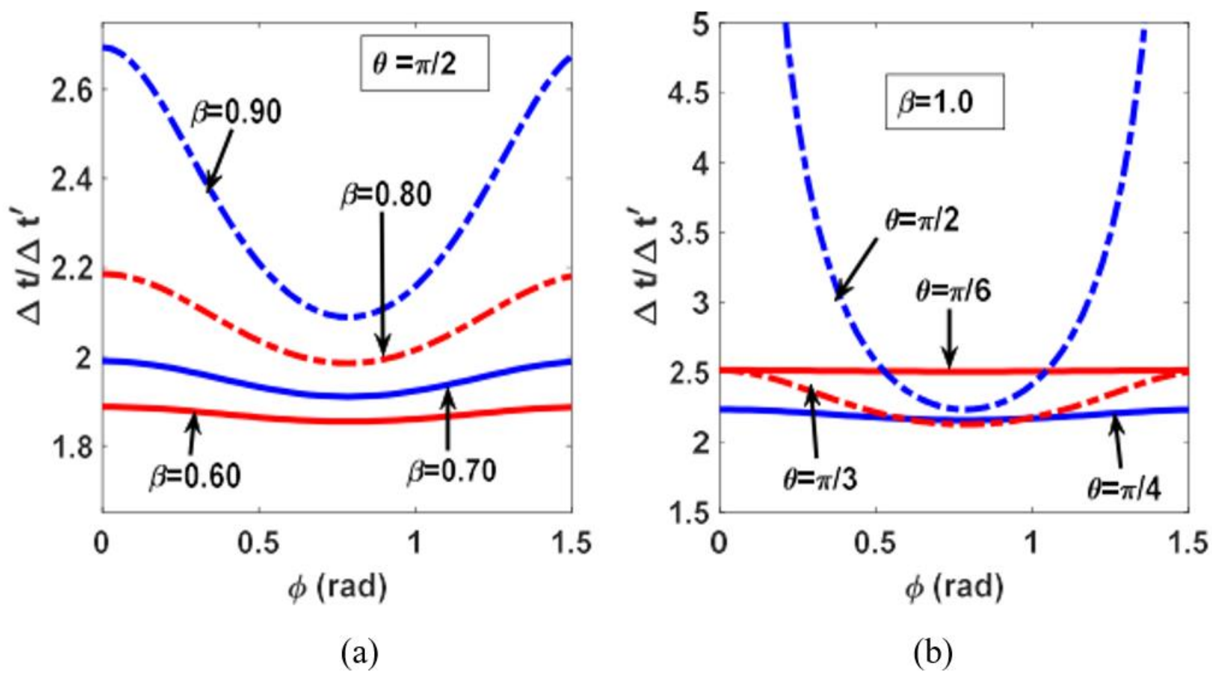


Figure 4. Polar plot of normalized Cartesian components of the time dilation as a function of azimuthal angle ϕ for $\beta=v/c=$ 0.60, 0.70, 0.80, 0.90 and $\theta=\pi/2$ (a) and $\beta=1$, $\theta=\pi/6,\pi/4,\pi/3,\pi/2$ (b), respectively.

4.2. Length Contraction

Since $dx_i^{\,2} = g_{x_i x_i} dx_i'^2$ (or $dx_i^{\,2} = \gamma_{x_i x_i}^2 dx_i'^2$) with $x_i = x, y, z$, the Cartesian components of the length contraction can be written as

$$\Delta l_x = \left(1 - \frac{u_x'^2}{c'^2}\right)^{-1/2} \Delta l_x'; \quad \Delta l_y = \left(1 - \frac{u_y'^2}{c'^2}\right)^{-1/2} \Delta l_y'; \quad \Delta l_z = \left(1 - \frac{u_z'^2}{c'^2}\right)^{-1/2} \Delta l_z' \tag{35}$$

Just like the time vector, the length of a rod moving with a three-dimensional velocity in both frames $l = \left(l_x^2 + l_y^2 + l_z^2\right)^{1/2}$ and $l' = \left(l_x'^2 + l_y'^2 + l_z'^2\right)^{1/2}$ is also measurable and can be written as:

$$\Delta l = \left(\Delta l_x^2 + \Delta l_y^2 + \Delta l_z^2\right)^{1/2} = \left(\frac{1}{1 - u_x'^2/c'^2} + \frac{1}{1 - u_y'^2/c'^2} + \frac{1}{1 - u_z'^2/c'^2}\right)^{1/2} \Delta l' \tag{36}$$

In one dimensional motion of $S_6^{'}$ frame parallel to + x axis of S_6 frame at constant speed smaller than speed of light (v<c), $\Delta l'$ length contraction of a rod, at rest in the $S_6^{'}$ frame and measured by the observer in S_6' frame is γ^{-1} with respect to Δl contraction of the rod, at rest and measured in S_6 frame.

5. RELATIVISTIC MASS, MOMENTUM AND ENERGY

Since Einstein's paper on special relativity [9] the concept of relativistic mass has been a topic of considerable experimental [22-24] and theoretical [25-28] interest over the years. In this section we will use the first postulate of special relativity [9] to derive the expression for relativistic mass of a free particle moving in massive frames. The differential increase in the energy of a free relativistic particle moving under the influence of a net force in the $S_6^{'}$ and S_6 frames are written as:

$$dE' = (\vec{F}'.\vec{u}')dt' = \vec{u}' \cdot d(m'\vec{u}') = \vec{u}' \cdot d\vec{p}' = \frac{1}{m'} \vec{p}' \cdot d\vec{p}' = c'^2 dm' \tag{37a}$$

$$dE = (\vec{F}.\vec{u})dt = \vec{u} \cdot d(m\vec{u}) = \vec{u} \cdot d\vec{p} = \frac{1}{m} \vec{p} \cdot d\vec{p} = c^2 dm \tag{37b}$$

where c' and c are speeds of light in the $S_6^{'}$ and S_6 massive frames and will be shown to be Lorentz scalars (having the same numerical value) in section 7. Equation 37a and 37b can be written in integral forms as

$$c'^2 \int_{m'(0)}^{m'(u')} dm' = \int_{p'(0)}^{p'(u')} u'dp' = \int_{m'(0)}^{m'(u')} u'^2 dm' + m' \int_{u'(0)}^{u'(c)} u'du' \tag{38a}$$

$$c^2 \int_{m(0)}^{m(u)} dm = \int_{p(0)}^{p(u)} udp = \int_{m(0)}^{m(u)} u^2 dm + m \int_{u(0)}^{u(c)} udu \tag{38b}$$

In the case of one-dimensional motion along +x axis, Equation 38a and 38b can be written as:

$$\int_{m'(0)}^{m'(u_x')} \frac{dm'}{m'} = \int_{u'(0)}^{u'(u_x')} \frac{du'}{c'^2 - u'^2}; \qquad \int_{m(0)}^{m(u_x)} \frac{dm}{m} = \int_{u(0)}^{u(u_x)} \frac{du}{c^2 - u^2} \tag{39}$$

Setting $c'^2 - u_x'^2 = \eta'^2$ $(c^2 - u_x^2 = \eta^2)$ with $u_x'(0) = c'$ $(u_x(0) = c)$ and $\sqrt{c'^2 - u_x'^2} = \eta'$ $(\sqrt{c^2 - u_x^2} = \eta)$ in the first (second) integral expressions in Equation 39 and integrating both sides, we find the following expressions for the relativistic mass of a free particle moving in the S_6' and S_6 frames

$$m'(u_x') = \frac{m'(0)}{\sqrt{1-u_x'^2/c'^2}}; \qquad m(u_x) = \frac{m(0)}{\sqrt{1-u_x^2/c^2}} \tag{40}$$

where $m'(0) = m_0$ $(m(0) = m_0)$ is rest mass and $u_x'(0) = 0$ $(u_x(0) = 0)$ initial velocity, respectively.

The relativistic mass which is anisotropic along the three axes in both massive frames can be written as

$$m_{xx}' = \frac{m_0}{\sqrt{1-u_x'^2/c^2}} = \gamma_{xx}' m_0; \qquad m_{yy}' = \frac{m_0}{\sqrt{1-u_y'^2/c^2}} = \gamma_{yy}' m_0; \qquad m_{zz}' = \frac{m_0}{\sqrt{1-u_z'^2/c^2}} = \gamma_{zz}' m_0 \tag{41a}$$

$$m_{xx} = \frac{m_0}{\sqrt{1-u_x^2/c^2}} = \gamma_{xx} m_0; \qquad m_{yy} = \frac{m_0}{\sqrt{1-u_y^2/c^2}} = \gamma_{yy} m_0; \qquad m_{zz} = \frac{m_0}{\sqrt{1-u_z^2/c^2}} = \gamma_{zz} m_0 \tag{41b}$$

Figure 5 shows the variation of anisotropy of m_{xx} and m_{yy} with azimuthal angle ϕ (with $0 \le \phi \le \pi/2$) for several $\beta = v/c$ ratios and polar angles $\theta = \pi/6, \pi/4, \pi/3, \pi/2$. m_{xx} (m_{yy}) decreases (increase) as the azimuthal angle ϕ increases $\phi = \pi/2$ for every $\beta = v/c$ ratio.

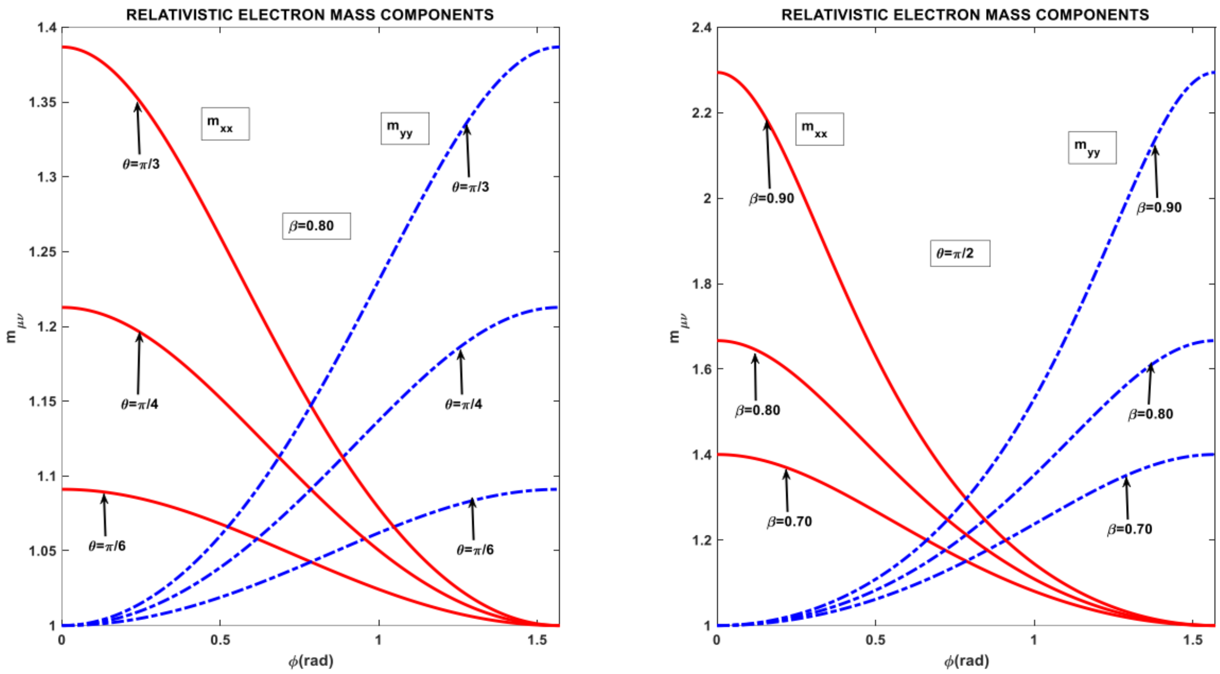


Figure 5. The angle variation of horizontal (red lines) and vertical (blue lines) components of normalized relativistic mass of electron for $\theta = \pi/6, \pi/4, \pi/3, \pi/2$ (a) and its magnitude for $\beta =$ 0.60, 0.70, 0.80 and 0.90 and $\theta = \pi/2$ when S_6' frame moves relative to S_6 frame.

Since $u_x^2 = u_x'^2$ and $c'^2 = c^2$ (See section 7.3), the relativistic mass is Lorentz scalar between two massive frames (e.g., $m'(u_x') = m(u_x)$). Equation 41a and 41b can be used to describe the linear momentum and energy of a relativistic particle moving in $S_6^{'}$ and S_6 frames:

$$\vec{p}' = m_x' u_x' \hat{i}' + m_y' u_y' \hat{j}' + m_z' u_z' \hat{k}' = \frac{m_0 u_x'}{\sqrt{1-u_x'^2/c'^2}}\hat{i}' + \frac{m_0 u_y'}{\sqrt{1-u_y'^2/c'^2}}\hat{j}' + \frac{m_0 u_z'}{\sqrt{1-u_z'^2/c'^2}}\hat{k}' \tag{42a}$$

$$\vec{p} = m_x u_x \hat{i} + m_y u_y \hat{j} + m_z u_z \hat{k} = \frac{m_0 u_x}{\sqrt{1-u_x^2/c^2}}\hat{i} + \frac{m_0 u_y}{\sqrt{1-u_y^2/c^2}}\hat{j} + \frac{m_0 u_z}{\sqrt{1-u_z^2/c^2}}\hat{k} \tag{42b}$$

Integrals of Equation 37a and 37b can also be written in the following ways:

$$\int_{p'(0)}^{p'(u')} p'dp' = c'^2 \int_{m'(0)}^{m'(u')} m'dm' \quad or \quad \frac{1}{m'}\int_{p'(0)}^{p'(u')} p'dp' = c'^2 \int_{m'(0)}^{m'(u')} dm' \tag{43a}$$

$$\int_{p(0)}^{p(u)} pdp = c^2 \int_{m(0)}^{m(u)} mdm \quad or \quad \frac{1}{m}\int_{p(0)}^{p(u)} pdp = c^2 \int_{m(0)}^{m(u)} dm \tag{43b}$$

Evaluating first integrals in Equation 43a and 43b, multiplying both sides of results by c'^2 and c^2, respectively, and then taking square root of final results yields the following expressions

$$E' = \hbar\omega' = m'c'^2 = \left(\frac{m_0^2 u_x'^2 c'^2}{1-u_x'^2/c'^2} + \frac{m_0^2 u_y'^2 c'^2}{1-u_y'^2/c'^2} + \frac{m_0^2 u_y'^2 c'^2}{1-u_y'^2/c'^2} + m_0^2 c'^4 \right)^{1/2} \tag{44a}$$

$$E = \hbar\omega = mc^2 = \left(\frac{m_0^2 u_x^2 c^2}{1-u_x^2/c^2} + \frac{m_0^2 u_y^2 c^2}{1-u_y^2/c^2} + \frac{m_0^2 u_z^2 c^2}{1-u_z^2/c^2} + m_0^2 c^4 \right)^{1/2} \tag{44b}$$

Which are the six-dimensional analogous of Einstein's energy dispersion relation [9]. Using Equation 44a and 44b, respectively, we can show that the product of group and phase velocities $v_g = d\omega'/dk' = c'$ and $u_p' = \omega'/k' = c'$ of electromagnetic waves moving in vacuum at speed c' satisfy $v_g' u_p' = c'^2$ in the $S_6^{'}$ massive frame and $v_g u_p = c^2$ in the S_6 massive frame at speed c. Evaluating the second integrals in Equation 43a and 43b, respectively, one finds the following equations for the energy dispersion relations

$$E' = \hbar\omega' = m'c'^2 = \frac{m_0 u_x'^2}{2\sqrt{1-u_x'^2/c'^2}} + \frac{m_0 u_y'^2}{2\sqrt{1-u_y'^2/c'^2}} + \frac{m_0 u_z'^2}{2\sqrt{1-u_z'^2/c'^2}} + m_0 c'^2 \tag{45a}$$

$$E = \hbar\omega = mc^2 = \frac{m_0 u_x^2}{2\sqrt{1-u_x^2/c^2}} + \frac{m_0 u_y^2}{2\sqrt{1-u_y^2/c^2}} + \frac{m_0 u_z^2}{2\sqrt{1-u_z^2/c^2}} + m_0 c^2 \tag{45b}$$

Similarly, using the energy dispersion relations given by Equation 45a and 45b we can show that $v_g' u_p' = c'^2$ in the $S_6^{'}$ massive frame and $v_g u_p = c^2$ in the S_6 massive frame at the constant speed.

6. RELATIVISTIC DOPPLER SHIFT AND DISPERSION RELATION

In the framework of classical four-dimensional theory of special relativity, the shift in classical Doppler effect formulae in relativistic regime is often given by the following equation [29].

$$\omega' = \gamma\left(1-\beta\cos\theta\right)\omega \tag{46}$$

where θ is the angle between the relative velocity of the reference frames and the direction of light propagation. Ives and Stilwell [30] observed the wavelength of hydrogen atom emitted by canal rays with and against their motion by using a mirror and discovered that the frequencies of displaced lines of incoming and outgoing rays and their average are given [29].

$$\omega'_{+} = \gamma\left(1-\beta\right)\omega; \qquad \omega'_{-} = \gamma\left(1+\beta\right)\omega; \qquad \omega'_{av} = \frac{\omega'_{+}+\omega'_{-}}{2} = \gamma\omega \tag{47}$$

In this section, we will consider plane waves of frequencies ω and ω' and wave vectors $\vec{k}$ and $\vec{k}'$ in the S_6 and $S_6^{'}$ massive frames, respectively, with the following wave functions

$$\varphi = Ae^{\vec{\omega}.\vec{t}-\vec{k}\cdot\vec{r}}; \qquad \varphi^{'} = A'e^{\vec{\omega}'.\vec{t}'-\vec{k}'\cdot\vec{r}'} \tag{48}$$

where $\omega/k = \omega'/k' = c$ in both massive frames. We write the angular frequencies as vectors $\vec{\omega}$ and $\vec{\omega}'$

$$\vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}, \qquad \vec{\omega}' = \omega'_x\hat{i}' + \omega'_y\hat{j}' + \omega'_z\hat{k}' \tag{49}$$

where x, y, and z- components of $\vec{\omega}$ and $\vec{\omega}'$ in the spherical coordinates are written as

$$\omega_x = \omega\sin\theta\cos\phi, \qquad \omega_y = \omega\sin\theta\sin\phi, \qquad \omega_z = \omega\cos\theta \tag{50a}$$

$$\omega'_x = \omega'\sin\theta\cos\phi, \qquad \omega'_y = \omega'\sin\theta\sin\phi, \qquad \omega'_z = \omega'\cos\theta \tag{50b}$$

Since phases of plane wave in S_6 and $S_6^{'}$ frames are Lorentz invariant ($\Delta\phi = 0$), we write

$$\vec{\omega}.\vec{t} + \vec{k}\cdot\vec{r} = \vec{\omega}'.\vec{t}' + \vec{k}'\cdot\vec{r}'; \qquad \vec{\omega}.\vec{t} - \vec{k}\cdot\vec{r} = \vec{\omega}'.\vec{t}' - \vec{k}'\cdot\vec{r}' \tag{51}$$

Taking the dot products of $\vec{\omega}$ and $\vec{t}$ in S_6 and of $\vec{\omega}'$ and $\vec{t}'$ in $S_6^{'}$, one finds

$$\begin{aligned}\omega_{x,+}\left(1-\beta_x\right)t_x + \omega_{y,+}\left(1-\beta_y\right)t_y + \omega_{z,+}\left(1-\beta_z\right)t_z \\ = \omega^{'}_{x,+}\gamma_{t_xt_x}\left(1-\beta_x^2\right)t_x + \omega^{'}_{y,+}\gamma_{t_yt_y}\left(1-\beta_y^2\right)t_y + \omega^{'}_{z,+}\gamma_{t_zt_z}\left(1-\beta_z^2\right)t_z,\end{aligned} \tag{52a}$$

$$\begin{aligned}\omega_{x,-}\left(1+\beta_x\right)t_x + \omega_{y,-}\left(1+\beta_y\right)t_y + \omega_{z,-}\left(1+\beta_z\right)t_z \\ = \omega^{'}_{x,-}\gamma_{t_xt_x}\left(1+\beta_x^2\right)t_x + \omega^{'}_{y,-}\gamma_{t_yt_y}\left(1+\beta_y^2\right)t_y + \omega^{'}_{z,-}\gamma_{t_zt_z}\left(1+\beta_z^2\right)t_z\end{aligned} \tag{52b}$$

Where $\vec{k}'\cdot\vec{r}' = 0$, which can be proven by using $\omega/k = \omega'/k' = c$ and Equations 20a and 20b. Component by component matching the both sides of Equations 52a and 52b gives

$$\omega'_{x,+} = \frac{\left(1-\beta_x\right)}{\gamma_{xx}\left(1-\beta_x^{\ 2}\right)}\omega_{ox}, \qquad \omega^{'}_{y,+} = \frac{\left(1-\beta_y\right)}{\gamma_{yy}\left(1-\beta_y^{\ 2}\right)}\omega_{oy}, \qquad \omega^{'}_{z,+} = \frac{\left(1-\beta_z\right)}{\gamma_{yy}\left(1-\beta_z^{\ 2}\right)}\omega_{oz}, \tag{53a}$$

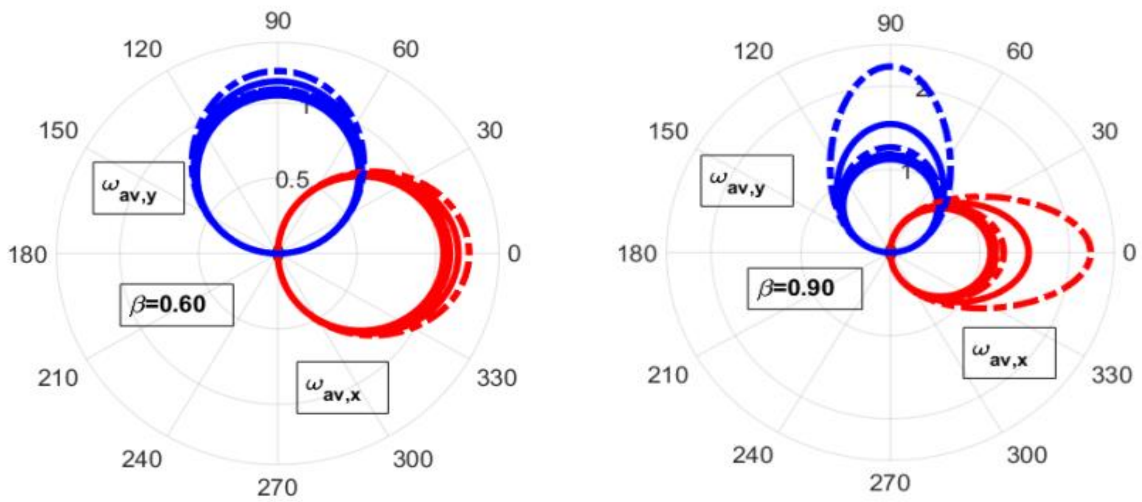
$$\omega'_{x,-} = \frac{(1+\beta_x)}{\gamma_{xx}(1-\beta_x{}^2)}\omega_{ox}, \qquad \omega'_{y,-} = \frac{(1+\beta_y)}{\gamma_{yy}(1-\beta_y{}^2)}\omega_{oy}, \qquad \omega'_{z,-} = \frac{(1+\beta_z)}{\gamma_{yy}(1-\beta_z{}^2)}\omega_{oz} \tag{53b}$$

Where ω_{ox}, ω_{oy}, and ω_{oz} are the components angular frequency in inertial frame. When $S_6^{'}$ moves parallel to x axes ($\phi = 0$) of S_6, $\omega'_{x,+}$ and $\omega'_{x,-}$ reduce to their one dimensional analogous ω'_+ and ω'_- in Equation 47. Using the Cartesian components of $\vec{\omega}'$, we write their averages in the $S_6^{'}$ frame as

$$\omega'_{av,x} = \frac{\omega'_{x+} + \omega'_{x-}}{2} = \gamma_{xx}\omega_{ox}; \qquad \omega'_{av,y} = \frac{\omega'_{y+} + \omega'_{y-}}{2} = \gamma_{yy}\omega_{oy}; \qquad \omega'_{av,z} = \frac{\omega'_{z+} + \omega'_{z-}}{2} = \gamma_{zz}\omega_{oz} \tag{54}$$

Where $\omega'_{av,x}$ reduces to its one-dimensional analogue ω'_{av} in Equation 47. Figure 6 shows the variation of the relativistic Doppler shifts of the average frequencies as a function of ϕ.

Figure 6a suggests that $\omega_{av,x}$ and $\omega_{av,y}$ are nearly circular for speeds less than speed of light (e.g., $\beta = 0.60$ for $\theta = \pi/2$) in vacuum. Meanwhile, Figure 6b indicates that $\omega_{av,x}$ and $\omega_{av,y}$ are nearly elliptical at speeds close to speed of light (e.g., $\beta = 0.90$) for $\theta = \pi/2$).



(a) (b)

Figure 6. Horizontal and vertical components of the average angular frequency of forward and reverse plane waves as a function of azimuthal angle ($0 \le \phi \le 2\pi$) with several polar angles $\theta = \pi/6, \pi/4, \pi/3, \pi/2$ for $\beta = v/c = 0.60$ (a) and $\beta = v/c = 0.90$ (b), respectively.

At this junction, we can show that there is a strong relation between Doppler shift and energy dispersion relation in massive frames. We write the angular frequency of plane wave in the $S_6^{'}$ frame as sum of the background frequency ω'_0 and Doppler shift $\Delta\omega'$, respectively.

$$\omega' = \omega_0' + \Delta\omega' = \omega_0' + \frac{1}{2}\omega_{av}' = \left(\omega_{0,x}' + \frac{1}{2}\omega_{av,x}'\right) + \left(\omega_{0,y}' + \frac{1}{2}\omega_{av,y}'\right) + \left(\omega_{0,z}' + \frac{1}{2}\omega_{av,z}'\right) \quad (55)$$

Where $\omega_0' = (\omega_{0,x}' + \omega_{0,y}' + \omega_{0,z}')$ is the absolute frequency at rest when $S_6^{'}$ massive frame coincides with a common inertial reference frame at $t' = t = 0$. Similar expression can be written in the S_6 massive frame. Using the classical relation $\omega = 2\pi / T$ between the period of the motion and angular frequency, one can use Equation 34 for the measurable time dilation to write the following expression for the angular frequency of plane wave in the $S_6^{'}$ massive frame.

$$\begin{aligned} \omega' &= \omega_{0,av}' + \frac{1}{2}\Delta\omega_{av}' = \omega_{0,av}' + \frac{1}{2}\left(\Delta\omega_{av,x}'^2 + \Delta\omega_{av,y}'^2 + \Delta\omega_{av,z}'^2\right)^{1/2} \\ &= \omega_{0,av}' + \frac{1}{2}\left(\frac{1}{1-u_x'^2/c'^2} + \frac{1}{1-u_y'^2/c'^2} + \frac{1}{1-u_z'^2/c'^2}\right)^{1/2} \Delta\omega_{0,x_i}' \end{aligned} \quad (56)$$

Where $\Delta\omega_{0,x_i}' = (\omega_{0,x}' + \omega_{0,y}' + \omega_{0,z}')$ is the shift in the absolute angular frequency of the plane wave in the $S_6^{'}$ massive frame which is coincident with a common inertial reference frame at $t' = t_0 = 0$. Similar expression can be written for the relativistic shift in the S_6 massive frame. This suggests a strong correlation between the measurable time dilation and Doppler shift, as expected.

Multiplying both sides of Equation 55 with $\hbar$, *or* Equation 56 *after applying the Binomial approximation to the square root term, with* $\hbar\Delta\omega_0' = (\hbar\Delta k_0')u' = \Delta p_0' u' = m_0' u'^2$ and $\hbar\Delta\omega_0 = (\hbar\Delta k_0)u = \Delta p_0 u = m_0 u^2$, *we* can write the following expressions for the energy dispersion relation of a relativistic particle moving *in* the $S_6^{'}$ and S_6 massive frames

$$E' = m'c'^2 = \frac{m_0 u_x'^2}{2\sqrt{1-u_x'^2/c'^2}} + \frac{m_0 u_y'^2}{2\sqrt{1-u_y'^2/c'^2}} + \frac{m_0 u_z'^2}{2\sqrt{1-u_z'^2/c'^2}} + m_0 c'^2 \quad (57)$$

Where $E_0' = \hbar\omega_0' = \hbar(\omega_{0,x}' + \omega_{0,y}' + \omega_{0,z}') = m_0 c'^2$ is the rest energies of relativistic particles in the $S_6^{'}$ massive frames when its origin is coincident with that of a common inertial frame at $t' = t = 0$. Similar expression can be written for the dispersion relation in the S_6 massive frame. Equation 57 equivalent to Equation 45a proving the validity of using $(1/2)\omega_{av}'$ in Equation 55 for relativistic shifts in the average angular frequencies of plane waves in vacuum. This suggest a strong relation between the time dilation and Doppler shift in the angular frequencies of plane waves and consequently, between Doppler shift and energy dispersion relation for the relativistic particles *moving* in the $S_6^{'}$ and S_6 massive frames, respectively.

7. RELATIVISTIC POWER AND INVARIANCE OF FELECTRIC AND MAGNETIC FIELDS

In the frame of the classical four dimensional spacetime theory, the velocity transformation Equations 10a and 10b are combined with the following force and power expressions to find the Lorentz invariance relations between Cartesian components of electric and magnetic fields in the S and S' frames [1].

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}; \qquad \vec{F}' = F'_x\hat{i} + F'_y\hat{j} + F'_z\hat{k} \tag{58}$$

$$\frac{dE}{dt} = \vec{F}.\vec{u} = F_x u_x + F_y u_y + F_z u_z\,; \qquad \frac{dE'}{dt'} = \vec{F}'.\vec{u}' = F'_x u'_x + F'_y u'_y + F'_z u'_z \tag{59}$$

Where dE/dt (dE'/dt') is the rate of relativistic energy $E = mc^2$ ($E' = m'c^2$) in the S (S') frame. $F_i = q(E_i + u_j B_k - u_k B_j)$ and $F'_i = q(E'_i + u'_j B'_k - u'_k B'_j)$ are the Cartesian components of Lorentz forces with u_j (u_k) and u'_j (u'_k) being x, y and z-components of $\vec{u}$ and $\vec{u}'$ in S and S' frames, respectively. Here, $i, j, k = x, y, z$ and q is the electric charge. Using Equations A1 and A2 one then writes x, y, z components of electric and magnetic fields in the S' (S) frame [1, 31].

$$\begin{aligned} &E'_x = E_x, \quad E'_y = \gamma\left(E_y - vB_z\right), \quad E'_z = \gamma\left(E_z + vB_y\right), \\ &B'_x = B_x, \quad B'_y = \gamma\left(B_y + (v/c^2)E_z\right), \quad B'_z = \gamma\left(B_z - (v/c^2)E_y\right) \end{aligned} \tag{60}$$

$$\begin{aligned} &E_x = E'_x, \quad E_y = \gamma\left(E'_y + vB'_z\right), \quad E_z = \gamma\left(E'_z - vB'_y\right), \\ &B_x = B'_x, \quad B_y = \gamma\left(B'_y - (v/c^2)E'_z\right), \quad B_z = \gamma\left(B'_z + (v/c^2)E'_y\right) \end{aligned} \tag{61}$$

Which state that only $E'_x = E_x$ and $B'_x = B_x$ are Lorentz invariant along the x-axis, but change occurs along y, and z-axes. Therefore, a purely electric (magnetic) field in the S' (S) frame is a mixture of the electric and magnetic fields in the S (S') frame, contrary to the common understanding in classical electrodynamics

In the frame of six dimensional spacetime theory proposed in this article, the conservation of relativistic power will be used to show that the electric and magnetic fields are Lorentz invariant between two massive frames. In doing so, we use the energy dispersion relations in Equations 37a and 37b, with zero rest mass, the following expressions are written for the relativistic power in the S'_6 and S_6 massive frames, respectively

$$\frac{dE'}{dt'_{x_i}} = \vec{F}'_{x_i}.\vec{u}'_{x_i} = F'_x u'_x + F'_y u'_y + F'_z u'_z \tag{62}$$

$$\frac{dE}{dt_{x_i}} = \vec{F}_{x_i}.\vec{u}_{x_i} = F_x u_x + F_y u_y + F_z u_z \tag{63}$$

Where $x'_i = x', y', z'$, $t'_{x_i} = t'_x, t'_y, t'_x$, $x_i = x, y, z$, $t_{x_i} = t_x, t_y, t_z$. Equations 62 and 63 are the six-dimensional analogues of the classical four-dimensional power relations in Equation A2. Suppose that the S'_6 and S_6 massive frames form a closed and isolated system, we can then apply the law of the conservation of relativistic power and write down the following equation:

$$dE'/dt'_{x_i} = dE/dt_{x_i} \quad \Rightarrow \quad \vec{F}'_{x_i} \cdot \vec{u}'_{x_i} = \vec{F}_{x_i} \cdot \vec{u}_{x_i} \tag{64}$$

Which shows that the relativistic power is invariant between the S_6' and S_6 massive frames under Lorentz transformation. Using Equations 29a and 29b for the velocity vectors in the S_6' and S_6 massive frames, respectively, one can rewrite Equation A6 in the following forms in both massive frames:

$$\begin{aligned} F_x u_x + F_y u_y + F_z u_z &= F_x' u_x' + F_y' u_y' + F_z' u_z' \\ &= \left(F_x' \cos\phi - F_y' \sin\phi\right) u_x + \left(F_x' \sin\phi + F_y' \cos\phi\right) u_y + F_z' u_z \end{aligned} \tag{65}$$

$$\begin{aligned} F_x' u_x' + F_y' u_y' + F_z' u_z' &= F_x u_x + F_y u_y + F_z u_z \\ &= \left(F_x \cos\phi + F_y \sin\phi\right) u_x' + \left(-F_x \sin\phi + F_y \cos\phi\right) u_y' + F_z' u_z' \end{aligned} \tag{66}$$

Which yield the following (3x3) matrix transformation for Cartesian components of $\vec{F}'$ in S_6' ($\vec{F}$ in S_6) massive frame expressed in terms of Cartesian components of $\vec{F}$ in S_6 ($\vec{F}'$ in S_6') massive frame:

$$\begin{pmatrix} F_x' \\ F_y' \\ F_z' \end{pmatrix}_{S_6'} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{S_6} ; \qquad \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}_{S_6} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x' \\ F_y' \\ F_z' \end{pmatrix}_{S_6'} \tag{67}$$

Matrix Equations 67 allow one to write the following expressions for net forces $\vec{F}'$ and $\vec{F}$ acting on the events observed in the S_6' frame, which takes place in the S_6 frame, respectively

$$\vec{F}' = F_x'\hat{i}' + F_y'\hat{j}' + F_z'\hat{k}' = \left(F_x \cos\phi + F_y \sin\phi\right)\hat{i}' + \left(-F_x \sin\phi + F_y \cos\phi\right)\hat{j}' + F_z\hat{k}' \tag{68}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = \left(F_x' \cos\phi - F_y' \sin\phi\right)\hat{i} + \left(F_x' \sin\phi + F_y' \cos\phi\right)\hat{j} + F_z'\hat{k} \tag{69}$$

Where $F_i = q(E_i + u_j B_k - u_k B_j)$ and $F_i' = q(E_i' + u_j' B_k' - u_k' B_j')$ are the Cartesian components of classical Lorentz forces in two frames, with $i, j, k = x, y, z$, which are written as

$$F_x' = q\left(E_x' + u_y' B_z' - u_z' B_y'\right), \quad F_y' = q\left(E_y' + u_z' B_x' - u_x' B_z'\right), \quad F_z' = q\left(E_z' + u_x' B_y' - u_y' B_x'\right) \tag{70}$$

$$F_x = q\left(E_x + u_y B_z - u_z B_y\right), \quad F_y = q\left(E_y + u_z B_x - u_x B_z\right), \quad F_z = q\left(E_z + u_x B_y - u_y B_x\right) \tag{71}$$

Substituting Cartesian components of Lorentz forces in Equations 70 and 71 into Equation 68 we can write

$$\left(E_x' + u_y' B_z' - u_z' B_y'\right) = \left(E_x + u_y B_z - u_z B_y\right)\cos\phi + \left(E_y + u_z B_x - u_x B_z\right)\sin\phi \tag{72}$$

$$\left(E_y' + u_z' B_x' - u_x' B_z'\right) = -\left(E_x + u_y B_z - u_z B_y\right)\sin\phi + \left(E_y + u_z B_x - u_x B_z\right)\cos\phi \tag{73}$$

$$E_z' + u_x' B_y' - u_y' B_x' = E_z + u_x B_y - u_y B_x \tag{74}$$

Using Equation 29b for the Cartesian components the u_x', u_y', and u_z' in Equations 72, 73 and 74 we find the following transformation matrix equations for the Cartesian components of the electric and magnetic fields in the S_6' frame in terms of those in the S_6 frame.

$$\begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}_{S'_6} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{S_6} ; \qquad \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix}_{S'_6} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{S_6} \tag{75}$$

The x, y and z components of magnetic field have a two-fold degeneracy in the S'_6 frame. Side by side addition of the Cartesian components of electric and magnetic fields gives:

$$\vec{E}' = E'_x\hat{i}' + E'_y\hat{j}' + E'_z\hat{k}' = \left(E_x\cos\phi + E_y\sin\phi\right)\hat{i}' + \left(-E_x\sin\phi + E_y\cos\phi\right)\hat{j}' + E_z\hat{k}' \tag{76}$$

$$\vec{B}' = B'_x\hat{i}' + B'_y\hat{j}' + B'_z\hat{k}' = \left(B_x\cos\phi + B_y\sin\phi\right)\hat{i}' + \left(-B_x\sin\phi + B_y\cos\phi\right)\hat{j}' + B_z\hat{k}' \tag{77}$$

Which show that the electric and magnetic fields are invariant along Cartesian coordinate axes between S'_6 and S_6 under forward Lorentz transformation. In other words, a pure electric (magnetic) field in S'_6 frame is composed of components of a pure electric (magnetic) field in S_6 frame.

In the case of inverse Lorentz transformation, substituting Cartesian components of Lorentz forces given by Equations 71 and 72 into Equation 69 and following the steps in writing Equations 72, 73 and 74, with the use of Equation 29a for the Cartesian components u_x, u_y and u_z of $\vec{u}$ in the S_6 frame, we write:

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}_{S_6} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \\ E'_z \end{pmatrix}_{S'_6} ; \qquad \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}_{S_6} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_x \\ B'_y \\ B'_z \end{pmatrix}_{S'_6} \tag{78}$$

Side by side addition of electric and magnetic field components in matric Equations 78 allow us to write the following equations for the electric and magnetic field vectors in the S_6 frame:

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k} = \left(E'_x\cos\phi - E'_y\sin\phi\right)\hat{i} + \left(E'_x\sin\phi + E'_y\cos\phi\right)\hat{j} + E'_z\hat{k} \tag{79}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = \left(B'_x\cos\phi - B'_y\sin\phi\right)\hat{i} + \left(B'_x\sin\phi + B'_y\cos\phi\right)\hat{j} + B'_z\hat{k} \tag{80}$$

which shows that Cartesian_components of electric and magnetic fields are invariant under the inverse Lorentz transformation between two massive frames. In other words, a pure electric (magnetic) field in the S'_6 massive frame is composed of components of a pure electric (magnetic) field in the S_6 massive frame.

8. RESULTS AND DISCUSSION

As pointed out in section 1, Equation 9 states that the spacetime coordinates in Voigt and Lorentz transformations differ from each other by a factor of Lorentz factor $\gamma = g_{\mu\nu}^{1/2}$ in the frame of four-dimensional spacetime and we can extend this similarity to six-dimensional space as $g'^{1/2}_{\mu\nu} = \gamma'_{\mu\nu}$ and $g_{\mu\nu}^{1/2} = \gamma_{\mu\nu}$ in S'_6 and S_6 frames, respectively. The effect of the anisotropy of six-dimensional Lorentz scaling factor on physical parameters such as relativistic mass, momentum and energy of moving particles can't easily be measured. However, we can explain the effect of the anisotropy of the six-dimensional Lorentz scaling factor on the various physical properties with the help of the idea proposed by Recami and Mignani [12]; Demers [13]; Mignani and Recami [14]; Cole [15]; Dattoli and Mingani [16]; Pappas [17]; Teli [18]; Guy [19]; Franco and Jorge [20]. Figure 7 shows the variation of γ_{xx}, γ_{yy}, and γ_{zz} with ϕ at several β values.

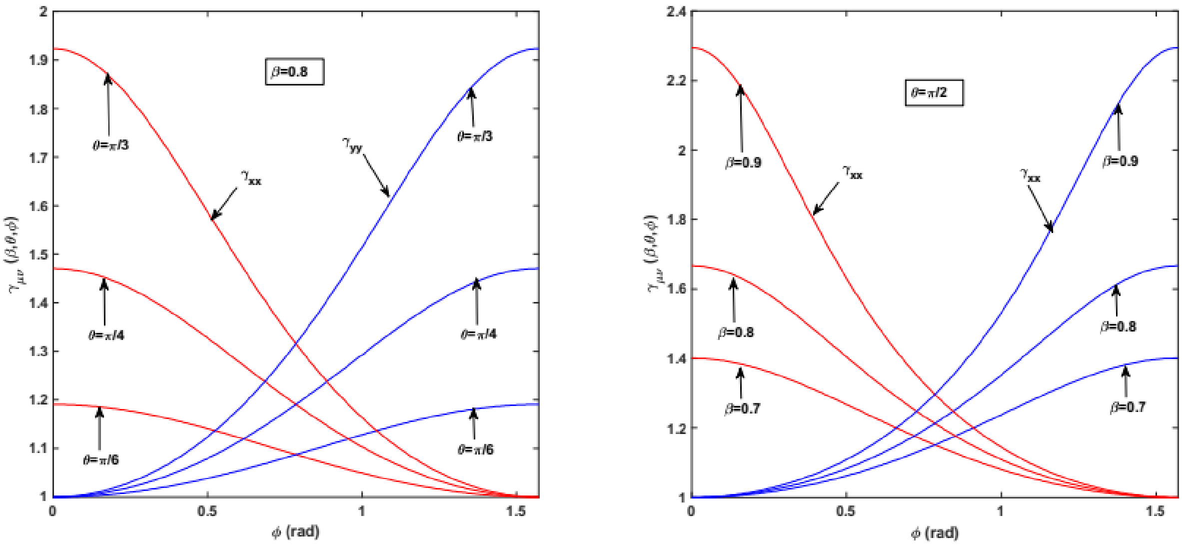


Figure 7. Angle variation of Lorentz scaling factors γ_{xx} (red line) and γ_{yy} (blue line) for (a) $\theta = \pi/6, \pi/4, \pi/3, \pi/2$ with $\beta = v/c = 0.8$ and (b) for $\beta = v/c$ =0.60, 0.70, 0.80 and 0.90 with $\theta = \pi/2$.

7.1. Relativistic Mass of a Particle

Since the relativistic energy of a free particle (e.g., photon) is $E = mc^2$, dividing both sides of Equation 44a with c'^2, the relativistic mass expressions in the $S_6^{'}$ massive frame is written as:

$$m' = \frac{E'}{c'^2} = \left(\frac{m_0^2 u_x'^2 / c'^2}{1 - u_x'^2 / c'^2} + \frac{m_0^2 u_y'^2 / c'^2}{1 - u_y'^2 / c'^2} + \frac{m_0^2 u_z'^2 / c'^2}{1 - u_z'^2 / c'^2} + m_0^2 \right)^{1/2} \tag{81}$$

With a expression written for the dispersion relation in the S_6 massive frame. Dividing both sides of Equation 44a with c'^2 the relativistic mass in the $S_6^{'}$ massive frame is written as

$$m' = \frac{E'}{c'^2} = \frac{m_0^2 u_x'^2 / c^2}{2\sqrt{1 - u_x'^2 / c^2}} + \frac{m_0^2 u_y'^2 / c^2}{2\sqrt{1 - u_y'^2 / c^2}} + \frac{m_0^2 u_z'^2 / c^2}{2\sqrt{1 - u_z'^2 / c^2}} + m_0 \tag{82}$$

With a expression written in the S_6 massive frame. Equation 81 and 82 yield $m'(0) = m(0) = m_0$ when particle is at rest. Predictions of Equation 40, 81, and 82 are compared with measured data for different β are given in Table 1 for the relativistic mass of electron.

Table 1. Measured and calculated relativistic electron mass are compared for different β ratios with $\phi = 0$ in cartesian coordinates.

v/c ratio	m/m₀ (Measured)	m_{xx}/m_0 Equation 40	m/m₀ Equation 81	m/m₀ Equation 82
0.317	1.059 (a)	1.055	1.052	1.053
0.429	1.106 (a)	1.107	1.097	1.102
0.506	1.188 (b)	1.156	1.139	1.149
0.634	1.298 (c)	1.293	1.233	1.260
0.688	1.370 (a)	1.378	1.285	1.326
0.696	1.404 (c)	1.393	1.294	1.338
0.707	1.426 (b)	1.402	1.299	1.344
0.750	1.507 (c)	1.511	1.359	1.425
0.801	1.6711 (b)	1.667	1.439	1.535

Note: (a): Bucherer [22] (b): Neumann [23] (c): Rogers, et al. [24].

The difference between calculated and measured mass is about 0.04% due to Equation 40 and about 4.2% due to Equation 81 and 82.

7.2. Rest Mass of a Relativistic Particle

In the framework of the classical theory of special relativity [9] one is required to substitute $m_0 = 0$ into the energy dispersion relation $E^2 = \hbar^2\omega^2 = c^2p^2 + m_0^2c^4$ to translate it into an energy-momentum relation $E = \hbar\omega = cp$ to satisfy Maxwell wave equations for electromagnetic waves travelling at constant speed of light in vacuum. However, the existence or nonexistence of the rest mass of photon in nature have been questioned by means of theoretical [32-40] and experimental [31, 41, 42] studies over the years. Recent experiments [31, 42] and observation [41] suggest that the photon rest mass is not zero, although its magnitude is small. In order to prove this view we use $E = \hbar\omega = mc^2$ and divide Equation 58 with c'^2 to write the rest mass as

$$m_0 = \left(\frac{u_x'^2/c^2}{1-u_x'^2/c^2} + \frac{u_y'^2/c^2}{1-u_y'^2/c^2} + \frac{u_z'^2/c^2}{1-u_z'^2/c^2} + 1\right)^{-1/2} \frac{\hbar\omega}{c^2} \tag{83}$$

Likewise, dividing both sides of Equation 59 with c'^2, we can write the rest mass as

$$m_0 = \left(\frac{u_x'^2}{2\sqrt{1-u_x'^2/c^2}} + \frac{u_y'^2}{2\sqrt{1-u_y'^2/c^2}} + \frac{u_z'^2}{2\sqrt{1-u_z'^2/c^2}} + 1\right)^{-1} \frac{\hbar\omega}{c^2} \tag{84}$$

Equation 83 and 84 suggest that m_0 is linear function of frequency at any azimuthal angle ϕ in spherical coordinates, with $m_0 = 0$ at *v=c* when one frame moves parallel to ±x axes ($\phi = 0, \pi$ and $\theta = \pi/2$) or parallel to ±y axes ($\phi = \pi/2$, $3\pi/2$) of another frame. It is gratifying to note that this is in excellent agreement with the Einstein's assumption of the photon rest mass at speed of light as two frames move in opposite directions relative to each other. Furthermore,

(i) $m_0 = \hbar\omega/c^2$ at *v=0* as its limiting case at any angle ($0 \le \phi \le 2\pi$ and $0 \le \theta \le \pi$).

The second result (ii) is compared with the prediction of Heisenberg uncertainty principle, which yields the following upper bound for the rest mass of photon in the inertial frame.

$$\Delta E \Delta t \geq \hbar \quad \to \quad m_0^{ub} = \hbar / c^2 T = (h/2\pi) f / c^2 = \hbar\omega / 2\pi c^2 \tag{85}$$

where $\Delta E = m_0^{ub} c^2$ is the rest energy and $\Delta t = T_0 = 1/f_0 \approx 13.80 Gyr$, is the age of the Universe [43].

It is interesting to note that the photon rest mass predicted by Equation 60 and 61 is 2π times higher than that predicted by Equation 62. When the azimuthal angle increases in counterclockwise direction.

from the horizontal x-axis $0 \leq \phi \leq \pi/2$ calculations suggest that $m_0 \neq 0$ for $\theta = \pi/6, \pi/4, \pi/3$.

Table 2 compares the nonzero photon rest mass calculated by using Equation 83 and 84 and 85 with the measured upper bound [31, 42] and observed data [41]. The dynamic torsion balance measurement of Luo, et al. [31] yields upper bound $m_0^{ub}(c) = 1.2 \times 10^{-54} kg$ for the photon mass at $f = 7.41 x 10^{-4} Hz$. Equation 60 and 61 give $m_0(c)$ = 4.34x10^{-55}kg at the same frequency. We also discover that the predicted rest mass 1.38×10^{-54} kg is in close agreement with the most recently observed photon mass 1.75x 10^{-53} kg at 2.36×10^{-3} Hz by Spallicci, et al. [41] who sought the deviation from Ampere-Maxwell law due to photon, through the NASA Magnetospheric Multiscale Mission (MMS) data for over six years. Calculated, measured and observed data suggest
that photon rest mass has a small magnitude but never zero.

Table 2. Comparison of calculated and measured rest mass of photon (m_0/kg).

f (Hz)	Equation 83 and 84	Equation 85	(Measured)
1.16x10^{-5}	4.28x10^{-56}	6.28x10^{-57}	2.0x10^{-53} [42]
7.41x10^{-4}	4.34x10^{-55}	6.91x10^{-56}	1.24x10^{-54} [31]
2.36 x10^{-3}	1.38x10^{-54}	2.20x10^{-55}	x10^{-53} [41]
2.30x10^{-18}	8.48x10^{-69}	1.35x10^{-69}	Uncertainty principle

7.3. Maxwell's Wave Equations with and without Photon Rest Mass

In this section, based on the Lorentz invariance of electric and magnetic fields given in section 7 , we will show that Maxwell's wave equations are invariant under the forward and inverse Lorentz transformations between two massive frames, we first need to prove the constancy of speed of light in all directions in both massive frames. We first write the Gauss laws of electrostatics and magnetostatics and Faraday's and Maxwell's laws of inductions [2] in the S_6 frame in vacuum.

$$\vec{\nabla} \cdot \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0; \quad \vec{\nabla}_i \times \vec{E} = -\frac{\partial \vec{B}}{\partial t_i}, \quad \vec{\nabla}_i \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t_i} \tag{86a}$$

$$\vec{E}(\vec{r}, t) = \vec{E}_0^{-i(\omega . t - \vec{k} . \vec{r})}; \qquad \vec{B}(\vec{r}, t) = \vec{B}_0^{-i(\omega . t - \vec{k} . \vec{r})} \tag{86b}$$

Where $i, j, k = x, y, z$ axes, respectively. Similarly, we write the Gauss laws of electrostatics and magnetostatics and Faraday's and Maxwell's laws of induction in the $S_6^{'}$ massive frame in vacuum.

$$\vec{\nabla}' \cdot \vec{E}' = 0, \quad \vec{\nabla}' \cdot \vec{B}' = 0; \quad \vec{\nabla}'_i \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t'_i}, \quad \vec{\nabla}'_i \times \vec{B}' = \frac{1}{c^2} \frac{\partial \vec{E}'}{\partial t'_i} \tag{87a}$$

$$\vec{E}'(\vec{r}',t') = \vec{E}'_0\, e^{-i(\omega'.t'-\vec{k}'.\vec{r}')}; \qquad \vec{B}'(\vec{r}',t') = \vec{B}'_0\, e^{-i(\omega'.t'-\vec{k}'.\vec{r}')} \tag{87b}$$

In proving the constancy of the speed of light in both massive frames, we insert the electric and magnetic fields into Faraday's and Maxwell's laws of inductions in one dimensional form and write the following equations.

$$-kE_0{}^{-i(\vec{\omega}.\vec{t}-\vec{k}.\vec{r})} = -\omega B_0{}^{-i(\vec{\omega}.\vec{t}-\vec{k}.\vec{r})} \quad \rightarrow \quad \frac{E_0}{B_0} = \frac{\omega}{k} = c \tag{88a}$$

$$-kB_0{}^{-i(\vec{\omega}.\vec{t}-\vec{k}.\vec{r})} = -\mu_0\varepsilon_0\omega E_0{}^{-i(\vec{\omega}.\vec{t}-\vec{k}.\vec{r})} \quad \rightarrow \quad \frac{E_0}{B_0} = \frac{k/\omega}{\mu_0\varepsilon_0} = \frac{1}{c\mu_0\varepsilon_0} \tag{88b}$$

Combining Equation 88a and 88b one finds $c = (\mu_0\varepsilon_0)^{1/2} = 3\times10^8 m/s$ in the S_6 frame. Furthermore, for $\vec{p} = \hbar\vec{k} = m\vec{u}$, $E = mc^2$, combination of Equation 65a and 65b gives

$$\frac{\omega^2}{k^2} = \frac{\hbar^2\omega^2}{\hbar^2k^2} = \frac{E^2}{p^2} = \frac{m^2c^4}{m^2u^2} = \frac{1}{\mu_0\varepsilon_0} \tag{89}$$

Which also gives $c = (\mu_0\varepsilon_0)^{1/2} = 3\times10^8 m/s$ for $u = c$. Furthermore, Equation 89 allows us to write the following one-dimensional energy dispersion relation in the S_6 frame

$$E^2 = \hbar^2\omega^2 = m^2c^4 = p_x^2c^2 + m_0^2c^4 = \gamma_x^2 m_0^2 u_x^2 c^2 + m_0^2c^4 \tag{90}$$

Where $\gamma_x = 1/(1-u_x^2/c^2)^{1/2}$ is equivalent to $\gamma = (1-v^2/c^2)^{1/2}$ in Equations 2a and 2b.

In the second step, inserting electric and magnetic fields given by Equation 87b into Equation 87a, for Faraday's and Maxwell's laws of inductions in one dimensional form, we find.

$$-k'\vec{E}'_0\, e^{-i(\vec{\omega}'.\vec{t}'-\vec{k}'.\vec{r}')} = -\omega' B'_0\, e^{-i(\vec{\omega}'.\vec{t}'-\vec{k}'.\vec{r}')} \quad \rightarrow \quad \frac{E'_0}{B'_0} = \frac{\omega'}{k'} = c' \tag{91a}$$

$$-k'B'_0\, e^{-i(\vec{\omega}'.\vec{t}'-\vec{k}'.\vec{r}')} = -\mu_0\varepsilon_0\omega' E'_0\, e^{-i(\vec{\omega}'.\vec{t}'-\vec{k}'.\vec{r}')} \quad \rightarrow \quad \frac{E'_0}{B'_0} = \frac{k'/\omega'}{\mu_0\varepsilon_0} = \frac{1}{c'\mu_0\varepsilon_0} \tag{91b}$$

Combining Equation 91a and 91b one finds $c' = (\mu'_0\varepsilon'_0)^{1/2} = 3\times10^8 m/s$ in the $S_6^{'}$ frame. Furthermore, using $\vec{p}' = \hbar\vec{k}' = m'\vec{u}'$ and $E' = m'c'^2$, combination of Equation 91a and 91b gives

$$\frac{\omega'^2}{k'^2} = \frac{\hbar^2\omega'^2}{\hbar^2k'^2} = \frac{E'^2}{p'^2} = \frac{m'^2c'^4}{m'^2u'^2} = \frac{1}{\mu_0\varepsilon_0} \tag{92}$$

which gives $c' = (\mu'_0\varepsilon'_0)^{1/2} = 3\times10^8 m/s$ For $u' = c'$ in the $S_6^{'}$ frame. Furthermore, Equation 69 allows us to write the following one-dimensional energy dispersion relation in the S'_6 frame

$$E'^2 = \hbar^2\omega'^2 = m'^2c'^4 = p_x'^2c'^2 + m_0^2c^4 = \gamma_x'^2 m_0^2 u_x'^2 c'^2 + m_0^2 c'^4 \quad (93)$$

where $\gamma_x' = 1/(1-u_x'^2/c'^2)^{1/2}$. We then conclude that the speed of light is Lorentz scalar

$c' = c = 3\times10^8 m/s$ in both frames with or without nonzero photon rest mass.

We are now ready to demonstrate that Maxwell's electromagnetic wave equations are invariant under Lorentz transformation between two massive frames in the frame of the 6-dimensional spacetime theory.

Applying the vector identity $\vec{\nabla}\times\vec{\nabla}\times = \vec{\nabla}\vec{\nabla}\cdot - \vec{\nabla}^2$ [2] to Faraday's and Maxwell's laws of inductions, we write

Maxwell's wave equations in S_6 and $S_6^{'}$ massive frames:

$$-\vec{\nabla}_i^{\,2}\vec{E} = -\frac{\partial}{\partial t_i}\left(\vec{\nabla}_i\times\vec{B}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t_i^2}; \qquad -\vec{\nabla}_i^{\,2}\vec{B} = -\frac{\partial}{\partial t_i}\left(\vec{\nabla}_i\times\vec{E}\right) = -\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t_i^2} \quad (94a)$$

$$-\vec{\nabla}_i'^2\vec{E}' = -\frac{\partial}{\partial t_i'}\left(\vec{\nabla}_i'\times\vec{B}'\right) = -\frac{1}{c^2}\frac{\partial^2\vec{E}'}{\partial t_i'^2}; \qquad -\vec{\nabla}_i'^2\vec{B}' = -\frac{\partial}{\partial t_i'}\left(\vec{\nabla}_i'\times\vec{E}'\right) = -\frac{1}{c^2}\frac{\partial^2\vec{B}'}{\partial t_i'^2} \quad (94b)$$

Where $\vec{\nabla}_i(\vec{\nabla}_i\cdot\vec{E}) = 0$, $\vec{\nabla}_i'(\vec{\nabla}_i'\cdot\vec{E}') = 0$, $\vec{\nabla}_i(\vec{\nabla}_i\cdot\vec{B}) = 0$, and $\vec{\nabla}_i'(\vec{\nabla}_i'\cdot\vec{B}') = 0$. Using chain rules for differential operators of x and t_x in Equations 20a and 20b and related ones in Equation 94a and 94b, we write

$$\begin{aligned}\nabla^2 E - \frac{1}{c^2}\left(\frac{\partial^2 E}{\partial t_x^2} + \frac{\partial^2 E}{\partial t_y^2} + \frac{\partial^2 E}{\partial t_z^2}\right) &= \nabla'^2 E' - \frac{1}{c^2}\left(\frac{\partial^2 E'}{\partial t_x'^2} + \frac{\partial^2 E'}{\partial t_y'^2} + \frac{\partial^2 E'}{\partial t_z'^2}\right) \\ &= a_{xx}\frac{\partial^2 E'}{\partial x'^2} + a_{yy}\frac{\partial^2 E'}{\partial y'^2} + a_{zz}\frac{\partial^2 E'}{\partial z'^2} - \frac{1}{c^2}\left(a_{t_xt_x}\frac{\partial^2 E'}{\partial t_x'^2} + a_{t_yt_y}\frac{\partial^2 E'}{\partial t_y'^2} + a_{t_zt_z}\frac{\partial^2 E'}{\partial t_z'^2}\right)\end{aligned} \quad (95a)$$

$$\begin{aligned}\nabla^2 B - \frac{1}{c^2}\left(\frac{\partial^2 B}{\partial t_x^2} + \frac{\partial^2 B}{\partial t_y^2} + \frac{\partial^2 B}{\partial t_z^2}\right) &= \nabla'^2 B' - \frac{1}{c^2}\left(\frac{\partial^2 B'}{\partial t_x'^2} + \frac{\partial^2 B'}{\partial t_y'^2} + \frac{\partial^2 B'}{\partial t_z'^2}\right) \\ &= a_{xx}\frac{\partial^2 B'}{\partial x'^2} + a_{yy}\frac{\partial^2 B'}{\partial y'^2} + a_{zz}\frac{\partial^2 B'}{\partial z'^2} - \frac{1}{c^2}\left(a_{t_xt_x}\frac{\partial^2 B'}{\partial t_x'^2} + a_{t_yt_y}\frac{\partial^2 B'}{\partial t_y'^2} + a_{t_zt_z}\frac{\partial^2 B'}{\partial t_z'^2}\right)\end{aligned} \quad (95b)$$

where $a_{\mu\nu}$ are coefficients given by Equation 18. Component by component matching of both sides of Equations 95a and 95b gives $a_{xx} = 1$, $a_{yy} = 1$, $a_{zz} = 1$ $a_{t_xt_x} = 1$, $a_{t_yt_y} = 1$, and $a_{t_zt_z} = 1$. Therefore, the covariant.

Maxwell wave equations become invariant under Lorentz transformation between two massive frames, which yields the Cartesian components of Lorentz scaling factor in Equation 19.

One can easily extend the invariance condition of Maxwell's wave equations in vacuum to materials medium by replacing c and c' with $c_m = 1/\sqrt{\mu_m\varepsilon_m}$ and $c_m' = 1/\sqrt{\mu_m'\varepsilon_m'}$ in S_6 and $S_6^{'}$ frames, respectively, in Equations 95a and 95b with the anisotropic Lorentz scaling factors.

$$\gamma_{m,xx} = \gamma_{m,t_x t_x} = 1/\sqrt{1-\beta_{mx}^2}, \quad \gamma_{m,yy} = \gamma_{m,t_y t_y} = 1/\sqrt{1-\beta_{my}^2}, \quad \gamma_{m,zz} = \gamma_{m,t_z t_z} = 1/\sqrt{1-\beta_{mz}^2} \tag{96}$$

Where $\beta_{mx} = v_x / c_m$, $\beta_{my} = v_y / c_m$, $\beta_{mz} = v_z / c_m$ are the normalized x, y, and z-components of relative velocities of S_6 and $S_6^{'}$ frames in material medium. Here $\varepsilon_m(\varepsilon_m')$ and $\mu_m(\mu_m')$ are the Lorentz scalar dielectric constant and magnetic permittivity of material medium in both frames.

9. CONCLISIONS

In this work extended the classical four- dimensional Minkowski spacetime theory to six dimensions by adding two extra time coordinates, which allows spatial time (position) change in position (time) in three coordinate axes and still satisfy the covariance and invariance conditions of the metric and Maxwell's wave equations between two frames under Voigt and Lorentz transformations, respectively. We introduce a new velocity transformation rule which is valid at any relative speed of reference frames moving with respect to each other. We derived expressions for relativistic mass, energy, time dilation, length contraction, Doppler shift, and photon rest mass, and Lorentz invariance of Maxwell wave equations between two reference frames in vacuum and materials medium. The predicted nonzero photon rest mass 4.34×10^{-53} kg at $f = 7.41 \times 10^{-4}$ Hz is in good agreement with measured upper bound 1.24×10^{-54} kg due to rotation torsion balance technique of Luo, et al. [31]. The calculated photon rest mass 1.38×10^{-54} kg at 2.36×10^{-3} Hz is also in close agreement with the most recently observed mass 1.75x 10^{-53} kg by Spallicci, et al. [41] who sought the deviation from Ampere-Maxwell law due to photon, through the NASA Magnetospheric Multiscale Mission (MMS) data for over six years. Calculated, measured and observed data suggest that photon rest mass has a small magnitude. But never zero. Furthermore, we also show that Maxwell's wave equations are form invariant under Lorentz transformation between two massive frames with and without nonzero photon mass in vacuum and in material medium.

Funding: This study received no specific financial support.
Competing Interests: The author declares that there are no conflicts of interests regarding the publication of this paper.

REFERENCES

[1] A. Einstein, "On the electrodynamics of moving bodies," *Annals of Physics*, vol. 17, pp. 891-921, 1905.

[2] D. J. Jackson, *Classical electrodynamics*, 3rd ed. New York: John Wiley & Sons, Inc, 1999.

[3] H. A. Lorentz, "Attempt at a theory of electrical and optical phenomena in moving bodies (E. J. Brill, Leiden). Retrieved from https://en.wikisource.org/wiki/Translation:AttemptofaTheoryofElectricalandOpticalPhenomenainMovingBodies," 1895.

[4] W. Voigt, "Ueber das doppler'sche princip," *Nachr Ges Wiss*, vol. 8, pp. 41-51, 1887.

[5] J. C. Freeman, "A new investigation into an unresolved mathematical procedure in the 1905 paper on special relativity," *Physics Essays*, vol. 31, pp. 310-322, 2018.

[6] A. G. Gluckman, "Coordinate transformations of W. Voigt and the principle of special relativity," *American Journal of Physics*, vol. 36, pp. 226-231, 1968.

[7] C. J. Masreliez, "Special relativity and inertia in curved spacetime," *Advanced Studies in Theoretical Physics*, vol. 2, pp. 795-815, 2008.

[8] R. Heras, "Lorentz transformations and the wave equation," *European Journal of Physics*, vol. 37, p. 025603, 2016.

[9] A. Ernst and J. P. Hsu, "First proposal of the universal speed of light by Voigt in 1887," *Chinese Journal of Physics*, vol. 39, pp. 211-230, 2001.

[10] H. Ünlü, "Special relativity in six dimensions," presented at the 37th International Physics Congress of Turkish Physical Society, September 1-5, 2021, Bodrum-Turkey, 2021.

[11] H. Unlü, "Invariance of electric and magnetic fields in relativistic electrodynamics. Available at: https://doi.org/10.21203/rs.3.rs-1543691/v1," 2022.

[12] E. Recami and R. Mignani, "Classical theory of tachyons (special relativity extended to superluminal frames and objects)," *The Review of the Nuovo Cimento (1971-1977)*, vol. 4, pp. 209-290, 1974.

[13] P. Demers, "Symmetrization of space and time in Lorentz space C 3 with linear algebra, as applied to the trichromatic theory," *Canadian Journal of Physics*, vol. 53, pp. 1687-1688, 1975.

[14] R. Mignani and E. Recami, "Classical theory of tachyons (special relativity extended to superluminal frames and objects) " *The Review of the Nuovo Cimento*, vol. 4, pp. 209-290, 1974.

[15] E. Cole, "Superluminal transformations using either complex space-time or real space-time symmetry," *The New Cimento A (1965-1970)*, vol. 40, pp. 171-180, 1977.

[16] G. Dattoli and R. Mingani, "Formulation of electromagnetism in six dimensional spacetime," *Lettre to the Nuovo Cimento*, vol. 22, p. 65, 1978.

[17] P. Pappas, "Physics in six dimensions-an axiomatic formulation," *Nuovo Cimento Lettere*, vol. 22, pp. 601-607, 1978.

[18] M. Teli, "General Lorentz transformations in six-dimensional spacetime," *Physics Letters A*, vol. 122, pp. 447-450, 1987.

[19] B. Guy, "The duality of space and time and the theory of relativity," *Journal of New Energy*, vol. 6, pp. 46-71, 2002.

[20] R. Franco and A. Jorge, "Vectorial Lorentz transformations," *Electronic Journal of Theoretical Physics*, vol. 3, pp. 35-64, 2006.

[21] H. Goldstein, *Classical mechanics*, 2nd ed. Reading, MA: Addison Wesley, 1980.

[22] A. H. Bucherer, "On the influence of earth movement on the intensity of light," *Annals of Physics*, vol. 316, pp. 270-283, 1903.

[23] G. Neumann, "The inert mass of fast-moving electrons," *Annals of Physics*, vol. 350, pp. 529-579, 1914.

[24] M. M. Rogers, A. McReynolds, and F. Rogers Jr, "Determination of the masses and velocities of three radium B beta-particles the relativistic mass of the electron," *Physical Review*, vol. 57, pp. 379-383, 1940.

[25] L. B. Okun, "The concept of mass," *Physics Today*, vol. 42, pp. 31-36, 1989.

[26] L. B. Okun, "Mass versus relativistic and rest masses," *American Journal of Physics*, vol. 77, pp. 430-431, 2009.

[27] L. B. Okun, *Energy and mass in relativity theory*: World Scientific, 2009.

[28] E. Hecht, "Einstein on mass and energy," *American Journal of Physics*, vol. 77, pp. 799-806, 2009.

[29] P. G. Bergmann, *Introduction to the theory of relativity*. New York: Dover Publications, 1975.

[30] H. E. Ives and G. R. Stilwell, "An experimental study of the rate of a moving atomic clock," *JOSA*, vol. 28, pp. 215-226, 1938.

[31] J. Luo, L.-C. Tu, Z.-K. Hu, and E.-J. Luan, "New experimental limit on the photon rest mass with a rotating torsion balance," *Physical Review Letters*, vol. 90, p. 081801, 2003.

[32] L. Bass and E. Schrodinger, "Must the photon mass be zero?," *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, vol. 232, pp. 1-6, 1955.

[33] A. S. Goldhaber and M. M. Nieto, "Terrestrial and extraterrestrial limits on the photon mass," *Reviews of Modern Physics*, vol. 43, pp. 277-296, 1971.

[34] L. Davis Jr, A. S. Goldhaber, and M. M. Nieto, "Limit on the photon mass deduced from Pioneer-10 observations of Jupiter's magnetic field," *Physical Review Letters*, vol. 35, pp. 1402-1405, 1975.

[35] L.-C. Tu, J. Luo, and G. T. Gillies, "The mass of the photon," *Reports on Progress in Physics*, vol. 68, pp. 77-130, 2004.

[36] A. S. Goldhaber and M. M. Nieto, "Photon and graviton mass limits," *Reviews of Modern Physics*, vol. 82, pp. 939-979, 2010.

[37] G. Spavieri, J. Quintero, G. Gillies, and M. Rodriguez, "A survey of existing and proposed classical and quantum approaches to the photon mass," *The European Physical Journal D*, vol. 61, pp. 531-550, 2011.

[38] P. Robles and F. Claro, "Can there be massive photons? A pedagogical glance at the origin of mass," *European Journal of Physics*, vol. 33, pp. 1217-1226, 2012.

[39] A. Arbab, "On the refractive index and photon mass," *Optik*, vol. 127, pp. 6682-6687, 2016.

[40] S. Klinaku and N. Syla, "Voigt, Lorentz, and the Michelson experiment," *Physics Essays*, vol. 30, pp. 148-149, 2017.

[41] A. D. Spallicci, G. Sarracino, O. Randriamboarison, and J. A. Helayël-Neto, "Testing the Ampère-Maxwell law on the photon mass and Lorentz-Poincaré symmetry violation with MMS multi-spacecraft data," *arXiv e-prints*, p. arXiv: 2205.02487, 2022.Available at: https://doi.org/10.48550/arXiv.2205.02487.

[42] R. Lakes, "Experimental limits on the photon mass and cosmic magnetic vector potential," *Physical Review Letters*, vol. 80, pp. 1826-1829, 1998.

[43] Planck, "Planck collaboration: Cosmological parameters. Retrieved from https://sci.esa.int/web/planck," vol. 32 2018.