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Evaluation of bitcoin options with interest rate risk and systemic risk

Pao-Peng Hsu[1+]
Chiang-Hui Wang[2]

[1]*Department of Insurance and Finance Management, Chaoyang University of Technology, Taiwan.*
Email: penghp@cyut.edu.tw
[2]*School of Financial Technology, Ming Chuan University, Taipei, Taiwan.*
Email: mandyw@mail.mcu.edu.tw

(+ Corresponding author)

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ABSTRACT

This study introduces closed-form formulas for valuing European call options, assuming that Bitcoin follows a compound Poisson process. Additionally, instantaneous forward interest rates are considered in the Heath-Jarrow-Morton model, which includes a jump component. To address the impacts of systematic risk on Bitcoin price and interest rate, we model two stochastic processes using a correlated bivariate jump-diffusion model to capture individual jumps and systematic co-jumps. This study provides analytic formulas for pricing Bitcoin call options and zero-coupon bonds under the correlated jump-diffusion Heath-Jarrow-Morton model. Numerical analysis shows how co-jump intensity affects the prices of both zero-coupon bonds and Bitcoin call options. We specifically look at how these prices change in response to co-jump intensity across three different instantaneous forward rate term structures. The findings show that the prices of Bitcoin call options are contingent on the term structure types of zero-coupon bonds. In addition, the interaction of co-jump intensity and types of term structure also affects Bitcoin option prices. The practical significance of this study is to provide a comprehensive model to evaluate Bitcoin call options and enhance risk management strategies in the Bitcoin market when the Bitcoin market encounters changes in monetary policy or changes in macroeconomic conditions.

Contribution/ Originality: In our proposed model, the evaluation of Bitcoin call options concerns not only the jump risks of interest rates but also the co-jump risks of interest rates and Bitcoin. We also analyze ZCB and call option prices' sensitivity to co-jump intensity across three different term structures.

1. INTRODUCTION

The average US Consumer Price Index (CPI) from October 2021 to January 2024 was 5.4%, exceeding the US Federal Reserve's target. For this reason, the US Federal Reserve has raised interest rates to lower the inflation rate. Starting from 1.822% in February 2022, the highest 10-year U.S. Treasury yield is 4.926% up to now, and the frequency of extreme interest rate values (plus or minus one standard deviation from the average) is 31% from October 2021 to January 2024. Those indicate a significant increase in interest rate fluctuations in recent years. The huge fluctuation of macroeconomic factors causes systemic jump risk. Systemic jump risk refers to the risk that affects the overall financial market or specific industries, typically associated with macroeconomic factors or global events such as the COVID-19 pandemic [1] impacting the majority of assets in the market [2] including Bitcoin. Typical systemic risks include financial crises, political instability, natural disasters, currency devaluation, and macroeconomic recessions. The same principle applies to correlated systemic jump risks.

Market-wide information on economic fundamentals correlates with co-jumps. Arouri, et al. [3] show the co-jumps in international equality markets. Especially unexpected events, such as changes in macroeconomic conditions and the release of financial earnings, can cause asset prices to jump. Lahaye, et al. [4] have applied non-parametric statistics to extract information on jumps and co-jumps in response to macroeconomic news from data on exchange rates, stock index futures, and US bond futures. This approach suggests that macroeconomic news and policy shifts precipitate not only individual asset jumps but also simultaneous jumps across multiple assets, as empirically demonstrated [5-7]. Particularly, susceptible to co-jumps, especially during interest rate policy changes, are interest rates and other assets. As mentioned in the previous sentences, in response to a sharp rise in the US CPI, the US Federal Reserve raised interest rates seven times in 2022, leading to a current Federal Funds Rate ranging between 5.00% and 5.25% as of June 2023. This phenomenon, which continues to affect the US dollar index, interest rates, equities, and potentially cryptocurrencies, underscores the impetus for our study.

Table 1. Jumps and co-jumps of US dollar index, 5-year treasury bond interest rate, and Bitcoin.

Jump numbers	Index	r	BTC
Numbers	12	21	22
Co-jump numbers	Index & r	r & BTC	Index & BTC
Numbers	3	3	2

Note: 1. Index, r and BTC represents, respectively, US dollar index, 5-year treasury bond interest rate and Bitcoin.

During the current high-inflation crisis, the US Federal Reserve's monetary policy continues to affect the US dollar index, interest rates, and Bitcoin's price. These three variables have experienced not only significant jumps but also co-jumps over the past 2 years. As indicated in Table 1, we can determine the presence of a jump in an underlying asset by comparing the underlying asset's weekly return with the benchmark; the benchmark is defined as twice the standard deviation of the underlying asset's weekly return. If the underlying asset's weekly return exceeds the benchmark, it indicates a jump. From January 2020 to the end of April 2023, the US dollar index, the 5-year Treasury interest rate, and Bitcoin price each experienced 12, 21, and 22 jumps, respectively. Bitcoin and interest rates have had three co-jumps: one in May 2020 and two in March 2022. In March 2022, three underlying assets had a co-jump. These data indicate that interest rates and Bitcoin have exhibited co-jumps. In addition, Hsu, et al. [8] provide an avenue for examining its cyclical patterns and structural changes influenced by the US dollar index and show Bitcoin to have a higher correlation with interest rates and the US dollar.

Bitcoin has higher volatility than traditional financial assets [9] and because of this property, Bitcoin is likely to bubble [10, 11]. The end of 2022 marked a Bitcoin price crash due to the collapse of Terra/Luna and the bankruptcy of FTX, which is a cryptocurrency platform for spot and derivatives trading. These events highlight the importance of cryptocurrency options as a dependable means of hedging price risks. Moreover, with the sharp increase in Bitcoin's volume, the demand for Bitcoin options has also increased significantly. According to the above overview of the financial situation, Bitcoin, interest rates, and the US dollar are affected by federal monetary policy and experience jumps and co-jumps. Moreover, evaluating Bitcoin call options concerns not only the jump risks of interest rates but also the common jump risks of interest rates and Bitcoin. We present a comprehensive European call option valuation model that accounts for both underlying assets and interest rates through jump-diffusion processes, distinguishing between individual jumps and co-jumps.

2. LITERATURE REVIEW

As the scale of the Bitcoin market gradually expands, research on Bitcoin option evaluation has increased significantly [12, 13]. These studies usually mentioned the phenomenon of large fluctuations and sharp jumps in Bitcoin prices [14]. The Black-Scholes (BS) model for pricing the European option fails to capture some part of skewness, heavy tails, and volatility. These limitations lead to the assumption of a jump-diffusion process as the underlying process in the BS model. Empirical studies have demonstrated jump behavior in individual assets [15, 16]. Therefore, several researchers have incorporated a jump-diffusion model into option pricing, assuming a constant interest rate [17] or stochastic interest rates [9, 18]. Nonetheless, to better reflect actual market conditions, several studies employ stochastic interest rate under the jump-diffusion models to characterize interest rate jumps [19-21].

The expanding empirical literature on co-jumps is increasingly influencing the incorporation of co-jump risks into option valuation. Lian, et al. [22] propose a method for the valuation of options on two assets; in this method, one asset is exchanged for another under the assumption that two assets follow correlated bivariate jump-diffusion (CJD) models capturing both individual jumps and systematic co-jumps. Qu, et al. [23] present a framework of jump-diffusion models for price dynamics with stochastic price volatilities and stochastic jump intensities. This framework employs a bivariate shot-noise process to model both stochastic variance and intensity; and accounts for common jump occurrences and nonnegative jump-size distributions for analyses of path-dependent options. Han, et al. [24] evaluate European crude oil options with a stochastic interest rate without jump risk. In their model, crude oil prices and convenience yield have co-jump effects. However, studies that apply the CJD model assume a constant interest rate. We propose a general European call option valuation model that allows underlying assets and interest rates to follow jump-diffusion processes, where jumps for underlying assets and interest rates can be distinguished into individual jumps and co-jumps.

Diffusions associated with stochastic volatility and correlated jumps have been used to assess Bitcoin because the cryptocurrency is an outlier characterized by extremely high volatility and frequent jumps [25, 26]. The stronger relationship between the US dollar and higher frequent jumps in the Bitcoin price implies that the Bitcoin price, interest rates, and the US dollar are affected by the US Federal monetary policies and undergo jumps and co-jumps. Therefore, the evaluation of Bitcoin call options concerns not only the jump risks of interest rates but also the co-jump risks of interest rates and Bitcoin. Several studies have addressed options valuation under jump-diffusion processes because of the Bitcoin jump phenomenon Scaillet, et al. [27] and Chen and Huang [28]. Hilliard and Ngo [29] characterize jumps and positive convenience for Bitcoin price and develop a theoretical jump-diffusion model for options on underlying assets with convenience yield risk. However, these studies all assume a constant interest rate, which is a limitation of the BS model. Our proposed model allows for interest rates to be stochastic. Therefore, this study presents a framework where the instantaneous forward rate adheres to the Heath-Jarrow-Morton (HJM) model [30] combined with a Poisson process, whereas the Bitcoin price follows a compound Poisson process; moreover, both the instantaneous forward rate and Bitcoin price are assumed to exhibit co-jump behavior. To our knowledge, no study has priced the systematic risks (co-jumps) and stochastic interest rate simultaneously on European options and zero-coupon bonds.

This model derives closed-form formulas for calculating call option prices. Moreover, the model maintains complete analytical tractability, yielding numerically evaluated closed-form formulas for option prices with high accuracy and efficiency. Additionally, we present a basic framework that assumes that the zero-coupon bond (ZCB) price and underlying asset follow the correlated jump-diffusion Heath-Jarrow-Morton (CJD–HJM) model. Our analysis includes a numerical simulation to elucidate the effects of co-jump intensity on call option pricing. We conducted a sensitivity analysis of ZCB and call option prices, examining their responsiveness to co-jump intensity across three different term structures.

The remainder of this paper is as follows: Section 2 delineates the proposed framework, which postulates that the Bitcoin price and instantaneous forward rate, representing the jump components, conform to a compound Poisson and Poisson distributions, respectively, and exhibit co-jump behavior. Furthermore, the section presents the notations for the instantaneous forward rate and ZCB under the CJD–HJM model. Section 3 presents closed-form formulas for evaluating European call options. Section 4 details the numerical analysis. Finally, Section 5 provides the study's conclusion.

3. PRICING A EUROPEAN CALL OPTION UNDER THE CJD–HJM MODEL

3.1. Model Setting and Forward Measure Transform

Consider a continuous-time financial market with a finite time horizon $[0,T]$, we assume that an instantaneous forward rate $f(t,T)$ follows a Poisson distribution. The stochastic differential equation for this rate is given as follows:

$$f(t,T) = f(0,T) + \int_0^t \mu_f(s,T)ds + \int_0^t \sigma_f(s,T)dW_f(s) + \int_0^t \beta(s,T)[dN_f(s) - \lambda_f^* ds] \qquad (1)$$

Where $\mu_f(t,T)$ and $\sigma_f(t,T)$ are the drift and volatility functions of the forward rate, respectively. The term $W_f(t)$ denotes a standard Wiener process under the natural measure P, and N_f is a Poisson process with constant intensity λ_f^*. The jump size is $\beta(t,T)$.

By according the findings of the martingale condition holds. Under this condition, the risk-free ZCB process can be expressed under the risk-neutral measure Q as follows:

$$\frac{dP(t,T)}{P(t-,T)} = r(t)dt - \xi_2(t,T)dW_f^Q(t) + \left(e^{-\xi_3(t,T)} - 1\right)[dN_f(t) - \theta_J(t)dt]$$

Where $r(t)$ represents the interest rate; $\xi_2(t,T) = \int_t^T \sigma_f(t,s)ds$; $\xi_3(t,T) = \int_t^T \beta(t,s)ds$

$$\text{and} \quad \theta_J(t) = \frac{\lambda_f^*}{e^{-\xi_3(t,T)}} \qquad (2)$$

In addition, the ZCB process is modeled as a compound Poisson process. We then proceed to construct a CJD–HJM model that incorporates underlying asset and ZCB price dynamics.

Let $(\Omega, \mathcal{F}, Q)$ denote a risk-neutral filtered probability space. The evolutions of Bitcoin price and ZCB under the jump-diffusion model and under the measure Q are expressed as follows:

$$\frac{dB(t)}{B(t-)} = r(t)dt + \sigma_B dW_B^Q(t) + (e^{U_B(t)} - 1)dN_B(t) \qquad (3)$$

$$\frac{dP(t,T)}{P(t-,T)} = r(t)dt - \sigma_P(t,T)dW_P^Q(t) + (e^{U_P(t)} - 1)dN_P(t) \qquad (4)$$

Where $dW_B^Q(t)$ and $dW_P^Q(t)$ with correlation ρ are standard correlated Brownian motions on $(\Omega, \mathcal{F}, Q)$ and σ_B and $\sigma_P(t,T)$ are the volatility of the Bitcoin price and ZCB price, respectively. In addition, $\sigma_P(t,T) = \xi_2(t,T)$, $U_P(t,T) = \xi_3(t,T)$, $\lambda_f^* = \lambda_P^*$, $W_P^Q(t) = W_f^Q(t)$ and $dN_P(t) = dN_f(t) - \theta_J(t)dt$. Let $N_B(t)$ and $N_P(t)$ denote the Poisson process with intensity rate λ_i^*, $i = B, P$; the corresponding jump amplitudes are controlled by $U_B(t)$ and $U_P(t)$. Moreover, the random variables $W_i^Q(t)$, $N_i(t)$, and U_i for $i = B, P$ are all mutually independent. Furthermore, $U_i(t)$ and $U_i(s)$ are independently and identically distributed, and $U_i(t) \sim N(\gamma_i, \delta_i^2)$, $i = B, P$; the mean percentage jump size is expressed as follows:

$$\kappa_i = E\left(e^{U_i(t)} - 1\right) = exp(\gamma_i + \frac{1}{2}\delta_i) - 1. \qquad (5)$$

A correlated bivariate jump model is constructed using three independent Poisson processes denoted by $n_B(t)$, $n_P(t)$ and $n_c(t)$. The independent Poisson process $n_i(t)$ has intensity λ_i^*, and the corresponding discrete probability density function is expressed as follows:

$$\text{Prob}(n_i(t) = k) = \frac{(\lambda_i^* t)^k}{k!} exp(\lambda_i^* t), \text{ for } i = B, P. \qquad (6)$$

Let $N_i(t) = n_i(t) + n_c(t)$. Because the characteristic function of the sum of two independent random variables is the product of the characteristic functions of the individual random variable, it follows that $N_i(t) \sim Possion(\lambda_i + \lambda_c)$, $\forall i = B, P$. Specifically, two jump types exist, namely, an individual jump with the intensity λ_i and co-jumps with the arrival intensity λ_c; that is, $\lambda_i^* = \lambda_i + \lambda_c$. The joint probability density function for $N_B(t)$ and $N_P(t)$ can be expressed as follows:

$$prob\left(n_B(t)=k, n_P(t)=g\right)=\sum_{m=0}^{\min(k,g)} \exp(-(\lambda_B \lambda_P \lambda_c)t) \frac{\left(\lambda_B t\right)^{k-m}\left(\lambda_P t\right)^{g-m}\left(\lambda_c t\right)^{m}}{(k-m)!(g-m)!m!}. \quad (7)$$

Hence, Equation 3 and 4 can be rewritten as follows, respectively:

$$B(t) = B(0)\, exp\left\{(r(t) - (\lambda_B + \lambda_c)\kappa_B - \frac{1}{2}\sigma_B^2)\tau + \sigma_B W_B^Q(t) + \sum_{i=1}^{n_B(t)} J_B(t) + \sum_{i=1}^{n_c(t)} J_c(t)\right\}, \quad (8)$$

Where $\tau = T - t$, $J_B(t) = \ln U_B(t)$ and $J_P(t) = \ln U_P(t)$.

$$P(t,T) = P(0,T)\exp\left\{(r(t) - (\lambda_P + \lambda_c)\kappa_P - \frac{1}{2}\sigma_P^2(t,T))\tau + \sigma_P(t,T) W_P^Q(t) + \sum_{i=1}^{n_P(t)} J_P(t) + \sum_{i=1}^{n_c(t)} J_c(t)\right\} \quad (9)$$

In addition, Lemma 1 provides the closed formula for the ZCB price.

Lemma 1: Assume that the instantaneous forward rate processes follow the Poisson–HJM model; the ZCB prices at $t < T$ can be expressed as follows:

$$P(t,T) = \exp(A\ (t,T) - B\ (t,T) r(t)), \quad (10)$$

Where

$$A(t,T) = ln(\frac{P(0,T)}{P(0,t)}) + B(t,T)\left(f(0,t) + \int_0^t \partial_2 D(s,t) ds\right) - \int_0^t [D(s,T) - D(s,t)] ds;;$$

$$D(t,T) = \int_t^T M(t,s)\, ds;\ \sigma_f(t,s) = \bar{\sigma}_f e^{-K(s-t)};\ \beta(t,s) = \bar{\beta} e^{-K(s-t)}$$

$$\int_0^t M\ (s,t)\ ds = e^{-K\ t t} r(0) + \int_0^t e^{-K\ (t-s)} U\ (s) ds - f(0,t); \quad (11)$$

$$U\ (t) = \frac{\partial}{\partial t} f(0,t) + K\ ff(0,t) + \int_0^t \left(\bar{\sigma}_f e^{K\ (s-t)}\right)^2 ds + \left(\int_0^t \theta_J(s) (\bar{\beta} e^{K\ (s-t)})^2 e^{-\xi_3(s,t)} ds\right); \quad (12)$$

$$\xi_3(t,T) = \frac{\bar{\beta}}{K}\left[1 - e^{K\ (T-t)}\right];\ \theta_J(t) = \frac{\lambda_f^*}{e^{-\xi_3(t,T)}};\ B\ (t,T) = \frac{1}{\psi(t)} \int_t^T \psi(u) du; \quad (13)$$

$\psi(T) = -e^{K\ T}$; $\bar{\sigma}_f$, $\bar{\beta}$, and K are constant.

The co-jump effects on the ZCB price are captured by $\theta_J(s)$ in Equation 13. Specifically, these effects involved in Equations 11–13.

3.2. The Valuation of Bitcoin Option

We consider a European call option $C(t,T)$ with strike K and maturity at T. The option price at time t can be evaluated as follows:

$$\begin{aligned} C(t,T) &= E^Q\left[e^{-\int_t^T r(s)ds}(B(T)-K)^+ \middle| F_t\right] \\ &= B(t)\, prob^{Q_B}\left[B(T) > K\right] - KP(t,T)\, prob^{Q_T}\left[B(T) > K\right] \\ &= B(t)\, prob^{Q_B}\left[\frac{1}{F(t,T)} < \frac{1}{K}\right] - KP(t,T)\, prob^{Q_T}\left[F(t,T) > K\right]. \end{aligned} \quad (14)$$

Where Q_T represents the forward measure where the ZCB is selected as the numeraire and Q_B represents the reciprocal forward measure in which the Bitcoin price is selected as the numeraire. $F(t,T)=\frac{S(t)}{P(t,T)}$ defines the forward price. To evaluate Equation 14, we decompose the European call option $C(t,T)$ into two parts:

$$\text{Part I} = KP(t,T)prob^{Q_T}[F(t,T) > K]; \text{Part II} = B(t)prob^{Q_B}\left[\frac{1}{F(t,T)} < \frac{1}{K}\right]$$

Next, we change the appropriate numeraire. Similar to the method of numeraire change described by Han and Wang [31] when the risk-neutral measure Q transforms a forward measure Q_T or the risk-neutral measure Q transforms a reciprocal forward measure Q_B the parameters of co-jump risk have not changed.

The random Esscher transform $Q_T \sim Q$ and $Q_B \sim Q$ on F_t with parameters $(\bar{\varsigma}_B, \bar{\varsigma}_P)$ and $(\hat{\varsigma}_B, \hat{\varsigma}_P)$ are expressed as follows, respectively:

$$\frac{dQ_T}{dQ}|F_T = \frac{\exp\left(\int_0^T \bar{\varsigma}_B\sigma_B dW_B(t) + \int_0^T \bar{\varsigma}_P\sigma_P(t,T)dW_P(t)\right)}{E_Q\left[\exp\left(\int_0^T \bar{\varsigma}_B\sigma_B dW_B(t) + \int_0^T \bar{\varsigma}_P\sigma_P(t,T)dW_P(t)\right)|F_T\right]} \quad (15)$$

and
$$\frac{dQ_B}{dQ}|F_T = \frac{\exp\left(\int_0^T \hat{\varsigma}_B\sigma_B dW_B(t) + \int_0^T \hat{\varsigma}_P\sigma_P(t,T)dW_P(t)\right)}{E_{\square}\left[\exp\left(\int_0^T \hat{\varsigma}_B\sigma_B dW_B(t) + \int_0^T \hat{\varsigma}_P\sigma_P(t,T)dW_P(t)\right)|F_T\right]}. \quad (16)$$

Similar to Cheang and Teh [32] and by substituting Equation 8 and 9 into log-forward price and using the following equation:

$dW_B^{Q_T}(t) = -\rho\sigma_P dt + dW_B^Q(t)$ and $dW_P^{Q_T}(t) = -\sigma_P dt + dW_P^Q(t)$.

The log-forward price under the measure Q_T is given by

$$\log F(t,T) = \left[\left((\lambda_P+\lambda_c)\kappa_P - (\lambda_B+\lambda_c)\kappa_B\right) - \left(\frac{\sigma_P^2(t,T)}{2} + \frac{\sigma_B^2}{2}\right) + \rho\sigma_B\sigma_P(t,T)\right]dt$$
$$+\left(\sigma_B dW_B^{Q_T}(t) - \sigma_P(t,T)dW_P^{Q_T}(t)\right)$$
$$+\log(J_B(t)(dn_B(t)+dn_c(t))) - \log(J_P(t)(dn_P(t)+dn_c(t))). \quad (17)$$

Additionally, we assume that $W^{Q_T}(t) = \left[W_B^{Q_T}(t), W_P^{Q_T}(t)\right]'$ can be described as an instance of two-dimensional Brownian motion.

$$\Sigma^2(t,T) = \sigma_B^2 + \sigma_P^2(t,T) - 2\rho\sigma_B\sigma_P(t,T) \quad (18)$$

$\mathbf{W}^{Q_T}(t)$ can be transformed into a description of one-dimension Brownian motion $W^{Q_T}(t)$. By substituting Equation 18 into Equation 17, we can express the forward price under the measure Q_T as follows:

$$F(t,T) = F(0,T)\exp\left\{\theta(t,T)t + \Sigma(t)dW^{Q_T}(t) + \sum_{i=1}^{N_B(t)} J_{B,i}(t) - \sum_{i=1}^{N_P(t)} J_{P,i}(t)\right\}$$

where $\theta(t,T) = \left[((\lambda_P+\lambda_c)\kappa_P - (\lambda_B+\lambda_c)\kappa_B) - \left(\frac{\sigma_P^2(t,T)}{2} + \frac{\sigma_B^2}{2}\right) + \rho\sigma_B\sigma_P(t,T) + \frac{1}{2}\Sigma^2(t,T)\right]$

Denotes $\Pi_{kg} = prob(N_B = k, N_P = g)$

$$= \sum_{m=0}^{\min(k,g)} \exp(-(\lambda_B\lambda_P\lambda_c)\tau)\frac{(\lambda_B\tau)^{k-m}(\lambda_P\tau)^{g-m}(\lambda_c\tau)^m}{(k-m)!(g-m)!m!}, \tau = T-t.$$

$$\text{Part I} = KP(t,T)prob^{Q_T}\left[F(0,T)\exp\left\{\theta(t,T)\tau + \Sigma(t)dW^{Q_T}(t) + \sum_{i=1}^{N_B(t)} J_{B,i}(t) - \sum_{i=1}^{N_P(t)} J_{P,i}(t)\right\}\right.$$
$$\left. > K \,|\, N_B(t)=k, N_P(t)=g\right]\cdot\sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} \quad (19)$$

$$= \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} KP(t,T) prob^{Q_T}\left[F(0,T)e^{\mu} > K\right].$$

Where $\mu = \theta(t,T)\tau + \Sigma(t)dW^{Q_T}(t) + \sum_{i=1}^{N_B(t)} J_{B,i}(t) - \sum_{i=1}^{N_P(t)} J_{P,i}(t)$. Let the conditions for the Poisson process be $n_B(t)$, $n_P(t)$, and $n_c(t)$; hence, the term μ in Equation 19 is normally distributed because the jump variables $J_B(t)$ and $J_P(t)$ are normally distributed after the log transformation. That is,

$$\mu \sim N(\theta(t,T)\tau + \gamma_B k - \gamma_B g, \Sigma^2(t)\tau + \delta_B k + \delta_P g). \tag{20}$$

Therefore, Part I can be rewritten as follows:

$$\text{Part I} = \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} KP(t,T)$$

$$\times prob^{Q_T}\left[F(0,T)\exp\left[\theta(t,T)\tau + \gamma_B k - \gamma_B g + \sqrt{\frac{\Sigma^2(t)t + \delta_B k + \delta_P g}{t}} W^{Q_T}(t)\right] > K\right]$$

$$= \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} P(t,T) KN(d_2), \tag{21}$$

Where

$$d_2 = \frac{\ln\left(\frac{B(t)}{P(t,T)K}\right) - (R(t) + \frac{1}{2}\Theta^2(t))(T-t)}{\Theta(t)\sqrt{(T-t)}}; \tag{22}$$

$$R(t) = \theta(t,T) + \frac{k(\gamma_B - \frac{\delta_B^2}{2})}{\tau} - \frac{g(\gamma_P + \frac{\delta_P^2}{2})}{\tau};$$

$$\Theta(t) = \Sigma^2(t) + \frac{k\delta_B^2 + g\delta_P^2}{\tau}.$$

Then, we define the reciprocal forward price, which is expressed as $\frac{1}{F(t,T)} = \frac{P(t,T)}{B(t)}$. According to Cheang and Teh [32]

$$dW_B^{\ T}(t) = -(\rho\sigma_P - \sigma_B)dt + dW_B^{\ B}(t) \text{ and } dW_P^{\ T}(t) = -(\sigma_P - \rho\sigma_B)dt + dW_P^{\ B}(t) \tag{23}$$

By applying the same approach and substituting Equation 23, we derive the reciprocal forward price process under the measure Q_B. Subsequently, Part II is calculated as follows:

$$\text{Part II} = \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} B(t) prob^{Q_B}\left[\frac{1}{F(t,T)} < \frac{1}{K}\right]$$

$$= \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} B(t) N(d_1) \tag{24}$$

Where

$$d_1 = \frac{\ln\left(\frac{B(t)}{P(t,T)K}\right) + (R(t) + \frac{1}{2}\Theta^2(t))(T-t)}{\Theta(t)\sqrt{(T-t)}}. \tag{25}$$

By substituting Lemma 1, Equation 21, and Equation 24 into $C(t,T)$, we can obtain Proposition 1 as follows:

Proposition 1. Under the CJD–HJM model, the closed-form Black–Scholes formula for a European Bitcoin call option can be expressed as follows:

$$C(t,T) = \sum_{k=0}^{\infty}\sum_{g=0}^{\infty}\Pi_{kg} P(t,T)\left[B(t)N(d_1) - KN(d_2)\right],$$

Where $P(t,T)$, d_1, and d_2 follow Equation 10, 25 and 23, respectively.

Proposition 1 indicates the effects of the interest rate and Bitcoin having a co-jump phenomenon on the call option. $\Theta(t)$ is a function of co-jumps with a specific arrival intensity and a function of individual jumps with a specific intensity. The intensity and magnitude of individual jumps can affect the call price of $\Theta(t)$ and $R(t)$, respectively. Furthermore, the call price formula is affected by the ZCB price when the interest rate is stochastic and by the jump intensity of the instantaneous forward rate.

4. NUMERICAL ANALYSIS

In this section, we conduct a numerical simulation to analyze the Bitcoin option under the CJD–HJM model, examining ZCB prices and Bitcoin options with co-jumps across three different term structures. We consider three shapes of initial instantaneous forward rates (IFRs) $f(0,t)$: hump (type-1), linear (type-2), and upward (type-3) term structures and refer to them for three term-structure functions. These correspond to the following three term-structure functions (where $\tilde{T} = T - t$, $\tilde{T} \in [0,10]$):

$$f_1(\tilde{T}) = -0.00088(\tilde{T} - 5)^2 + 0.03;\ f_2(\tilde{T}) = -0.0005\tilde{T}^{1.2} + 0.03; \quad \text{and}$$
$$f_3(\tilde{T}) = 0.0015(\tilde{T} - 3)^2 - 0.0003\tilde{T}^{2.3} + 0.02.$$

The parameters used in the simulations are detailed in Table 2.

Table 2. Parameter values for numerical analysis simulations.

Parameters	Value	Parameters	Value	Parameters	Value
K	0.66	$r(0)$	0.025	γ_B	0.04
ρ	-0.5	κ_B	0.66	γ_P	0.05
$\bar{\sigma}_f$	0.8	κ_P	0.75	λ_P	1
σ_B	1.2	δ_P	0.45	λ_B	1.5
$\bar{\beta}$	0.2	δ_B	0.5	λ_C	0.6/1.2/1.8

4.1. ZCB Prices Under Three Term Structures with Co-Jump Intensity

Utilizing the parameter settings from Table 2 and applying them to Lemma1 and Proposition 1, we determine ZCB prices under type-1, type-2, and type-3 term structures with varying co-jump intensities. A co-jump intensity of λ_C = 0.6 serves as the benchmark. We observe the ZCB prices as co-jump intensity increases, first doubling from λ_C = 0.6 to λ_C = 1.2 and then tripling to 1.8. We compare the differences in ZCB prices at λ_C = 0.6 versus λ_C = 1.2 and 1.8.

Figure 1a illustrates the impacts of co-jump intensities on ZCB prices under type-1 term structure under the CJD–HJM model.

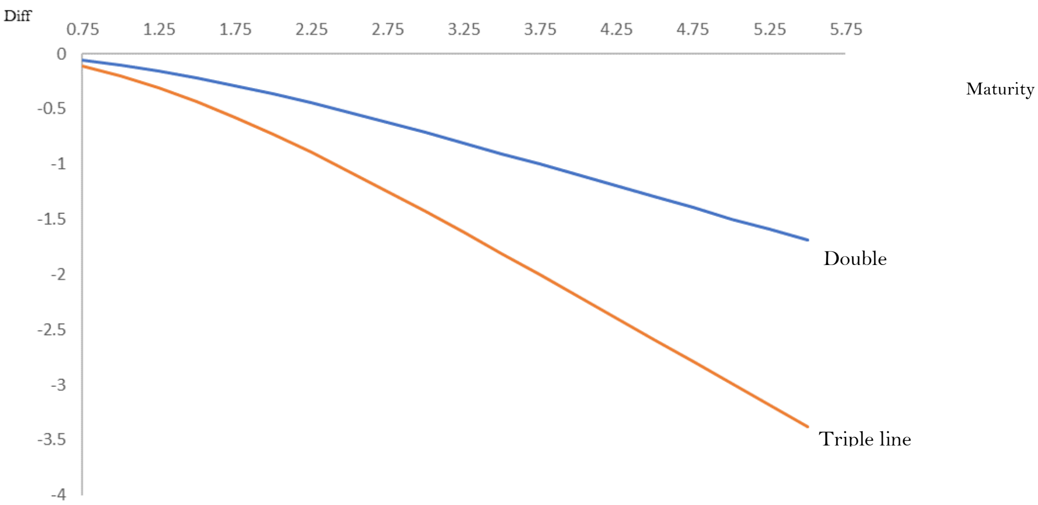

Figure 1a. ZCB prices under type-1 term structure under the CJD–HJM model.

Figure 1b illustrates the impacts of co-jump intensities on ZCB prices under type-2 term structure under the CJD–HJM model.

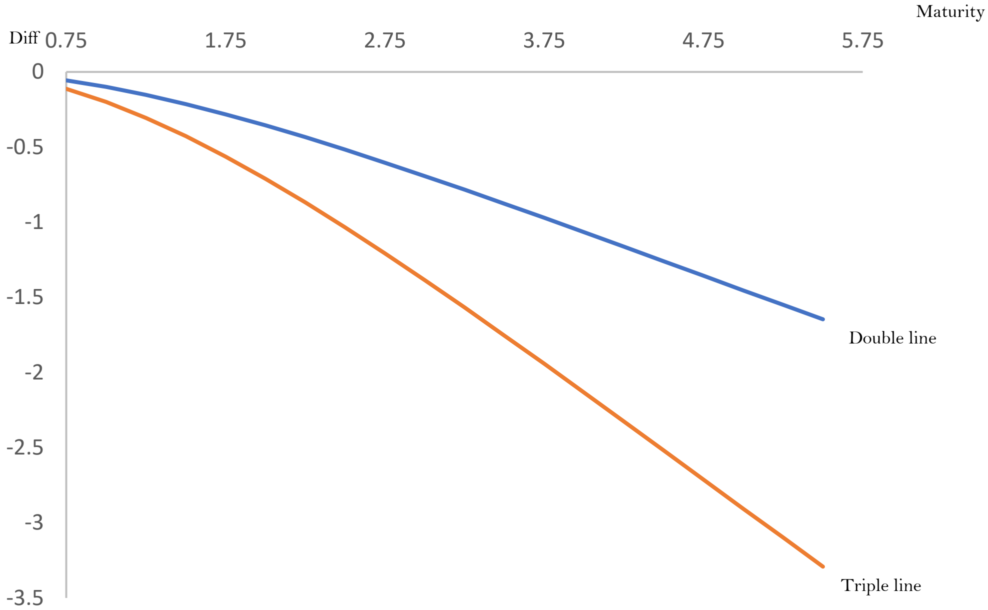

Figure 1b. ZCB prices under type-2 term structure under the CJD–HJM model.

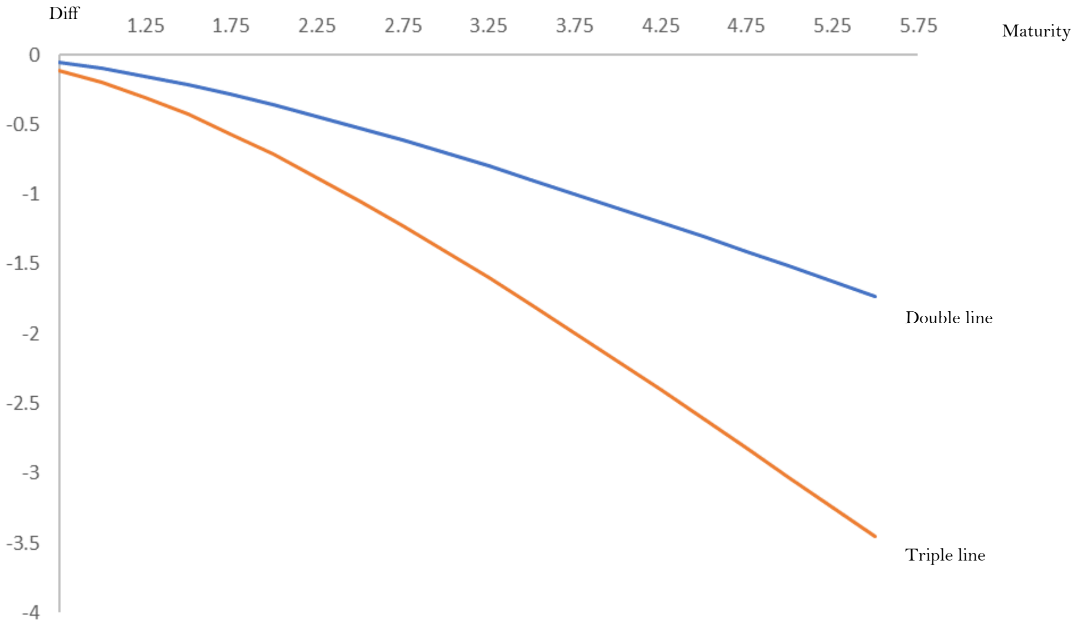

Figure 1c. ZCB prices under type-3 term structure under the CJD–HJM model.

Figure 1c illustrates the impacts of co-jump intensities on ZCB prices under type-3 term structure under.

Figure 1 displays the variation in ZCB prices as the function of double and triple co-jump intensities compared with the benchmark across three term structures. "diff" represents the variation in ZCB prices and "double line" and "triple line" represents, respectively, double and triple co-jump intensity. An increase in co-jump intensity, λ_C, is correlated with a reduction in ZCB prices. Consequently, a tripled λ_C results in a more substantial negative difference in ZCB prices than a doubled λ_C, regardless of the term structure. In addition, the disparity in ZCB prices widens with increasing maturity; for instance, the triple co-jump intensity is equal to −2.99, −2.90, and −3.03 for fiveyear maturities under type-1, type-2, and type-3 term structures, respectively. These results indicate that term structure type and co-jump intensity interact to affect ZCB pricing in the CJD–HJM model.

Figure 2a illustrates the impacts of co-jump intensities on call option prices under type-1 term structure in the CJD–HJM model.

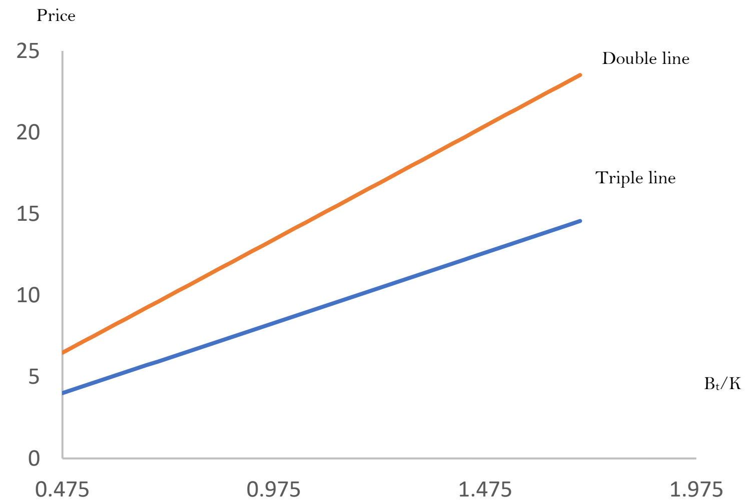

Figure 2a. Call prices under type-1 term structure in the CJD–HJM model.

Figure 2b illustrates the impacts of co-jump intensities on call option prices under type-2 term structure in the CJD–HJM model.

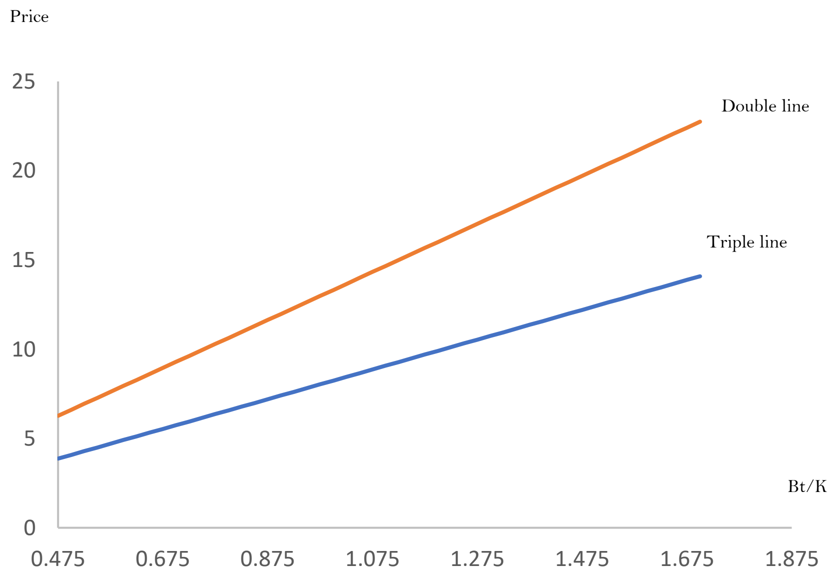

Figure 2b. Call prices under type-2 term structure in the CJD–HJM model.

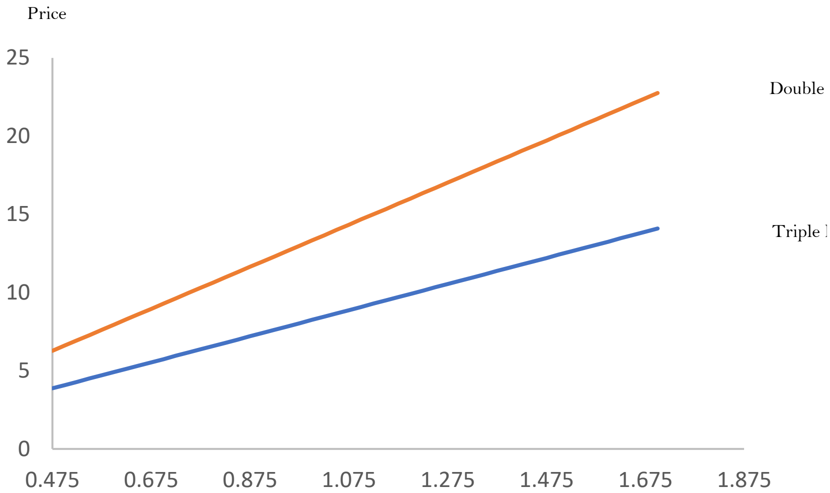

Figure 2c. Call prices under type-3 term structure in the CJD–HJM model.

Figure 2c illustrates the impacts of co-jump intensities on call option prices under type-3 term structure in the CJD–HJM model.

4.2. Bitcoin Call Option Prices under the CJD–HJM Model

Figure 2 displays the variation in ZCB prices as a function of double and triple co-jump intensities compared with the benchmark across three term structures. All settings are the same as in Figure 1. Each point on the triple line, regardless of term structures and whether it is in-the-money, or at-the-money, has a value that is smaller than the double line. They indicate that the greater the co-jump intensity, the lower the option price. The results imply that systematic risk reduces call option prices.

4.3. Sensitivity Analysis of Call Price under the CJD–HJM Model

Proposition 1 states that the term structure of ZCB prices influences the closed-form formula for the Bitcoin call option price, thereby making the Bitcoin call option price dependent on the type of ZCB. We define three call price types accordingly. Similarly, increase λ_C, by two to three times and consider the spread in call prices. As depicted in Figure 2, an increase in λ_C typically leads to a rise in call prices. Furthermore, the deeper an option is in the money, the wider the spread in call prices, indicating that co-jump intensity exerts a more substantial effect on deep-in-the-money call options than on deep-out-of-the-money call options. Call prices have a narrower spread in the triple line than in double line condition because of the corresponding decrease in ZCB prices. This variance underscores the critical difference between individual jump-diffusion and common jump-diffusion call prices under stochastic interest rates with jump processes. Accounting for the co-jump risk of interest rates and assets. Table 3 demonstrates the effects of instantaneous forward rate types on call price values.

Table 3. Sensitivity analysis of co-jump intensity to call prices.

B_t/K	Type-1		Type-2		Type-3	
	Triple	**Double**	**Triple**	**Double**	**Triple**	**Double**
Deep-out-of-the-money						
0.475	4.008	6.487	3.875	6.271	3.877	6.274
0.5	4.223	6.833	4.083	6.606	4.084	6.609
At-the-money						
1	8.524	13.775	8.241	13.318	8.245	13.323
Deep-in-the-money						
1.5	12.840	20.737	12.415	20.050	12.420	20.058
1.525	13.056	21.086	12.624	20.387	12.629	20.395

Table 3 presents a sensitivity analysis of co-jump intensity on call prices for deep-out-of-the-money, at-the-money, and deep-in-the-money scenarios. The call spreads in type-1 are the most significant across all cases. Call price spreads are comparable in type-2 and type-3 conditions.

5. CONCLUSION

This study has contributed significantly to the field of financial modeling by deriving closed-form prices for ZCBs and call options under the CJD-HJM model. We derived analytic formulas for ZCB's prices and call options under the CJD-HJM model by incorporating interest rate and systematic risk. Also, we have provided a comprehensive analysis of the effects of co-jump intensity on the pricing of ZCBs and call options across three distinct term structures. Our findings illustrate the relationship between co-jump risk, interest rate fluctuations, and Bitcoin option price.

Our study's findings emphasize the importance of considering co-jump intensity when pricing Bitcoin-related derivatives, particularly call options. We have demonstrated that co-jump intensity significantly impacts call prices, with deeper in-the-money options showing wider price spreads compared to deep out-of-the-money options. Those emphasize the necessity of incorporating co-jump risk into risk management strategies for exotic options, especially in high-inflation environments where systemic risks are prevalent. Moreover, our sensitivity analysis of call prices under the CJD-HJM model has revealed valuable insights into the dynamics of option pricing under different co-jump intensities and term structures. The variations in call prices across deep-out-of-the-money, at-the-money, and deep-in-the-money scenarios underscore the nuanced effects of co-jump intensity on option valuation. These findings give practitioners and researchers a deeper understanding of how co-jump risk influences option pricing and risk management strategies in volatile markets.

Our study offers valuable tools for managing co-jump risk and easily extends to pricing exotic options based on the CJD–HJM model, particularly in high-inflation environments. The framework can also be applied to value other underlying assets, particularly when the underlying asset is affected by interest rate changes or systematic co-jumps. Exploring the impact of co-jump risk on different derivatives and asset classes could provide valuable insights for risk management in financial markets. These findings contribute to the existing literature on exotic option pricing and set the stage for future financial modeling and risk management research.

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Competing Interests: The authors declare that they have no competing interests.
Authors' Contributions: Both authors contributed equally to the conception and design of the study. Both authors have read and agreed to the published version of the manuscript.

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