Value of Pi

Summary

Traditional methods of calculating the value of Pi are approximate. This article demonstrates that there is an accurate method for calculating its value to 16 significant digits based on the performance of a computer program in Turbo C 2.0 including formulas resulting from the combination of mathematical properties of the circle and the theorem of Pythagoras.

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Introduction

Pi is a number represented by the Greek letter π. It is the ratio of the circumference of a circle to its diameter. It is used in many areas and there are many laws that recognize its value which the normal law with expectancy μ and standard deviation σ which the probability density function is written:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/(2\sigma^2)}$$

Several methods for calculating the value of π have been proposed but have the disadvantage of being approximate. It is possible by running a program in Turbo C 2.0, using mathematical formulas derived from combining the properties of the circle and the Pythagorean Theorem, to obtain the precise value of π to 16 significant digits.

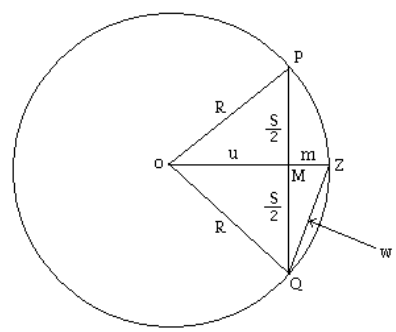
Review of the literature

The value of Pi is based on the eye such as:

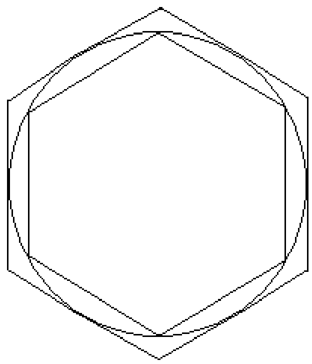
$$OM = u = \sqrt{R^2 - \left(\frac{S}{2}\right)^2}$$

$$MZ = m = R - u$$

$$ZQ = w = \sqrt{m^2 + \left(\frac{S}{2}\right)^2}$$



If n is the number of sides of a polygon inscribed in a circle of radius R, Liu Hui determined that since the scope of it is $p = nw$ $\pi = 3.14024$ for $n = 192$ and $R = 10$ and measuring S and recalculating, which is an excellent approximation factor of p likely to improve as n increases.



This method of approximation of the value of π is better than that of Archimedes earlier that the sides of regular polynomials, inscribed and circumscribed, to a circle of diameter 1 are numerous the more the value of π, represented by the ratio of the average perimeters of these to it, tends to its fair value between 3 and 3.47 on the basis of two hexagons and very close to 3.141592654 because the errors of measurements of these sides would be represented by human errors in their measures themselves and manufacturing errors of the instrument to measure them.

Many other methodologies of calculation such as the Chinese method of calculation by false position, that of continued fractions, jet needles, and Monte Carlo yield values very close to those obtained by Archimedes and Liu Hui and even that by a Chinese calculator Sharp EL-506R π =3.141592654

Mathematical Model

I am strict about the need to be more precise to calculate π, since I propose that if n_0=8, then:

$$p_0 = n_0 w_0 = n_0\sqrt{m^2 + \left(\frac{S_0}{2}\right)^2} = n_0\sqrt{(R-u)^2 + \left(\frac{S_0}{2}\right)^2}$$

$$= n_0\sqrt{\left[R - \sqrt{R^2 - \left(\frac{S_0}{2}\right)^2}\right]^2 + \left(\frac{S_0}{2}\right)^2}$$

$$= 2n_0 \frac{\sqrt{\left[R - \sqrt{R^2 - \left(\frac{S_0}{2}\right)^2}\right]^2 + \left(\frac{S_0}{2}\right)^2}}{2R} R = 2\pi R$$

where :

$$\pi_0 = \frac{n_0\sqrt{\left[R - \sqrt{R^2 - \left(\frac{S_0}{2}\right)^2}\right]^2 + \left(\frac{S_0}{2}\right)^2}}{2R} = \frac{n_0 w_0}{2R}$$

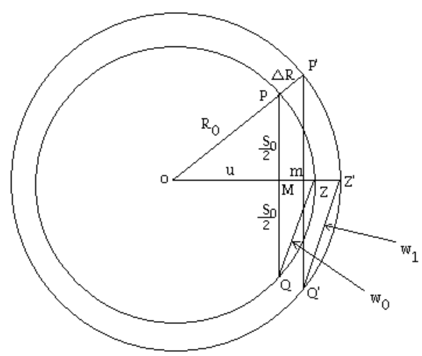
The rationale for this is that since OQZ and OQ'Z' are two similar triangles, if R_0=1 so if:

$$P_0 = n_0 w_0 = 2\pi$$

then:

$$\frac{w_1}{R_0 + \Delta R} = \frac{w_0}{R_0}$$

$$n_0 w_1 = \frac{R_0 + \Delta R}{R_0} n_0 w_0$$



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That is to say that the perimeter of the circle of radius R is:

$$p_1 = \frac{R_0 + \Delta R}{R_0} p_0 = R p_0, \quad R = R_0 + \Delta R = 1 + \Delta R$$

$$p_1 = 2\pi R$$

And the surface of the circle of radius R is S_1 such that:

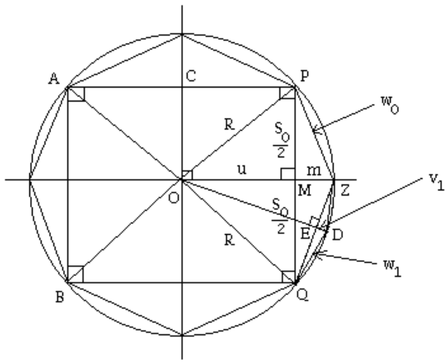
$$S_1 = \frac{n_1 w_1 R}{2} = \frac{p_1 R}{2} = \frac{2\pi R^2}{2} = \pi R^2$$

Among other arguments lack, all things being equal, is that if the perimeter of a circle of radius R is 2 π R, its surface is:

$$S = \int_0^R 2\pi R dR = 2\pi\left[\frac{R^2}{2}\right]_0^R = \pi R^2$$

And the area of a circle of radius 1 is π

The application of the Pythagorean Theorem by Liu Hui in the circle allows me to argue that



$$PZ^2 = MZ^2 + PM^2$$

that is to say

$$w_0^2 = \left[R - \sqrt{R^2 - \left(\frac{S_0}{2}\right)^2}\right]^2 + \left(\frac{S_0}{2}\right)^2$$

$$= R^2 - 2R\sqrt{R^2 - \left(\frac{S_0}{2}\right)^2} + R^2 - \left(\frac{S_0}{2}\right)^2 + \left(\frac{S_0}{2}\right)^2$$

$$2R\sqrt{R^2 - \left(\frac{S_0}{2}\right)^2} = 2R^2 - w_0^2$$

$$\sqrt{R^2 - \left(\frac{S_0}{2}\right)^2} = \frac{2R^2 - w_0^2}{2R}$$

$$R^2 - \left(\frac{S_0}{2}\right)^2 = \frac{\left(2R^2 - w_0^2\right)^2}{4R^2}$$

$$S_0 = 2\sqrt{R^2 - \frac{\left(2R^2 - w_0^2\right)^2}{4R^2}}$$

and it follows that if we stick to the above formulas π_0 can not be calculated after having measured w_0 and calculated S_0 or measured S_0 and calculated w_0 on the basis that $n_0=8$.

To overcome measurement errors of S_0 or w_0 and an inability to measure S_n or w_n and therefore to calculate the true $\pi = \pi_n$ when n is very high, it is necessary to apply the Pythagorean theorem differently to the circle of Liu Hui to make it clear that since ABPQ is a square with center O and OCPM is a second square

$$OM^2 + MQ^2 = R^2$$

that is to say:

$$\left(\frac{S_0}{2}\right)^2 + \left(\frac{S_0}{2}\right)^2 = R^2$$

$$2\left(\frac{S_0}{2}\right)^2 = R^2$$

$$S_0 = \frac{2R}{\sqrt{2}} = \sqrt{2}R$$

where S_0 is the true that is to say whose values dependent on R are at a number of significant digits higher than those of calculated S_0^* as:

$$S_0^* = 1.414213562R$$

by the calculator Sharp EL-506R and it follows that w_0, however, is smaller than R such as:

$$w_0 = \sqrt{\left[R - \sqrt{R^2 - \left(\frac{\sqrt{2}R}{2}\right)^2}\right]^2 + \left(\frac{\sqrt{2}R}{2}\right)^2} = \sqrt{\left(R - \sqrt{R^2 - \frac{R^2}{2}}\right)^2 + \frac{R^2}{2}}$$

$$= \sqrt{\left(R - \frac{R}{\sqrt{2}}\right)^2 + \frac{R^2}{2}} = \sqrt{\frac{\left(\sqrt{2}-1\right)^2}{2}R^2 + \frac{R^2}{2}}$$

$$= \sqrt{\frac{\left(\sqrt{2}-1\right)^2 + 1}{2}}R = \sqrt{\frac{4 - 2\sqrt{2}}{2}}R = \sqrt{2 - \sqrt{2}}R$$

$$w_0^* = 0.765366864R$$

by the same calculator for the minimum value of π that is π_0 such that:

$$\pi_0 = \frac{n_0 w_0}{2R} = \frac{8\sqrt{2-\sqrt{2}}}{2} = 4\sqrt{2-\sqrt{2}}$$

that is to say

$$\pi_0^* = 3.061467459$$

by the Sharp EL-506R calculator or

$$\pi_0^* = 3.0614674091339111$$

by running with the software program Turbo C 2.0 the program PI.C.

Moreover, since EZD is a rectangle triangle, applying again the Pythagorean Theorem allows:

$$ED^2 + EZ^2 = DZ^2$$

that is to say :

$$v_1^2 + \left(\frac{w_0}{2}\right)^2 = w_1^2$$

$$v_1 = \sqrt{w_1^2 - \left(\frac{w_0}{2}\right)^2}$$

and that

$$OE^2 + EZ^2 = OZ^2$$

that is to say:

$$(R - v_1)^2 + \left(\frac{w_0}{2}\right)^2 = R^2$$

$$\left[R - \sqrt{w_1^2 - \left(\frac{w_0}{2}\right)^2}\right]^2 + \left(\frac{w_0}{2}\right)^2 = R^2$$

$$R - \sqrt{w_1^2 - \left(\frac{w_0}{2}\right)^2} = \sqrt{R^2 - \left(\frac{w_0}{2}\right)^2}$$

$$\sqrt{w_1^2 - \left(\frac{w_0}{2}\right)^2} = R - \sqrt{R^2 - \left(\frac{w_0}{2}\right)^2}$$

$$w_1^2 - \left(\frac{w_0}{2}\right)^2 = \left[R - \sqrt{R^2 - \left(\frac{w_0}{2}\right)^2}\right]^2$$

$$w_1 = \sqrt{\left[R - \sqrt{R^2 - \left(\frac{w_0}{2}\right)^2}\right]^2 + \left(\frac{w_0}{2}\right)^2} = \sqrt{2R^2 - 2R\sqrt{R^2 - \left(\frac{w_0}{2}\right)^2}}$$

$$w_0 = \left[2 - \left(2 - \sqrt{2-\sqrt{2}}\right)\right]R$$

$$\frac{w_0}{2} = R - \left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)R$$

$$\left(\frac{w_0}{2}\right)^2 = R^2 - 2\left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)R^2 + \left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 R^2$$

$$R^2 - \left(\frac{w_0}{2}\right)^2 = 2\left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)R^2 - \left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 R^2$$

$$= -\left[\left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)^2 R^2 - 2\left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)R^2\right]$$

$$= -\left[\left(1 - \frac{\sqrt{2-\sqrt{2}}}{2}\right)R - R\right]^2 + R^2$$

$$= -\left(-\frac{\sqrt{2-\sqrt{2}}}{2}R\right)^2 + R^2$$

$$= -\frac{2-\sqrt{2}}{4}R^2 + R^2$$

$$= \left(1 - \frac{2-\sqrt{2}}{4}\right)R^2$$

Replacing this in the formula of calculation of w_1 allows it becomes:

$$w_1 = \sqrt{2R^2 - 2R\sqrt{\left(1 - \frac{2-\sqrt{2}}{4}\right)R^2}} = \sqrt{2R^2 - 2R^2\sqrt{1 - \frac{2-\sqrt{2}}{4}}}$$

$$= \sqrt{2 - 2\sqrt{1 - \frac{2-\sqrt{2}}{4}}}\,R = \sqrt{2 - \sqrt{4 - 2 + \sqrt{2}}}\,R = \sqrt{2 - \sqrt{2 + \sqrt{2}}}\,R$$

and it follows that:

$$\pi_1 = \frac{n_1 w_1}{2R} = \frac{16\sqrt{2 - \sqrt{2 + \sqrt{2}}}}{2}$$

$$\pi_1^* = 3.121445152$$

by the Sharp EL-506R calculator or

$$\pi_1^* = 3.1214447021484375$$

by running with the software program Turbo C 2.0 the program PI.C.

It follows by a recurrent reasoning that:

$$w_2 = \sqrt{2R^2 - 2R\sqrt{R^2 - \left(\frac{w_1}{2}\right)^2}}$$

$$w_3 = \sqrt{2R^2 - 2R\sqrt{R^2 - \left(\frac{w_2}{2}\right)^2}} \quad \ldots.$$

$$w_n = \sqrt{2R^2 - 2R\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}}$$

and that :

$$\pi_0 = \frac{8w_0}{2R} = \frac{4w_0}{R} = \frac{2^2 w_0}{R}$$

$$\pi_1 = \frac{2 \times 8w_1}{2R} = \frac{8w_1}{R} = \frac{2^3 w_0}{R}$$

$$\pi_2 = \frac{2 \times 2 \times 8w_2}{2R} = \frac{16w_1}{R} = \frac{2^4 w_0}{R} \quad \ldots.$$

$$\pi_n = \frac{2^n \times 8w_n}{2R} = \frac{2^{n+2} w_n}{R}$$

Program for calculating π

The program PI.C I designed for getting values of π with 16 decimal places is such that:

```
#include <stdio.h>
#include <math.h>
main ()
{
struct calpi
{
float w, w1, x1, x2, x3, x4, n, dn, pi;
};
struct calpi p [ψ]; /* put an integer in place of ψ the highest possible reflecting the large number of rows in the table of variables that is to say the number of iterations possible, therefore, of the extent of the pi calculable and their degree of precision*/
int i=1;
printf("\n\nPI.C calculate PI");
p[0].x1=2-sqrt(2);
p[0].w=sqrt(p[0].x1);
p[0].w1=p[0].w/2;
p[0].pi=4*p[0].w;
printf("\n\nn = 0");
printf("\n\nw0 = %.16f", p[0].w);
printf("\n\npi0 = %.16f", p[0].pi);
p[0].n=2;
do
{
p[i].x1=p[i-1].w/2;
p[i].x2=pow(p[i].x1,2);
p[i].x3=1-p[i].x2;
p[i].x4=2-2*sqrt(p[i].x3);
p[i].w=sqrt(p[i].x4);
p[i].w1=p[i-1].w/2;
p[i].n=p[i-1].n+1;
p[i].pi=pow(2,p[i].n)*p[i].w;
p[i].dn=p[i].w+p[i-1].w*(p[i].w-p[i-1].w)/(2*p[i].w
*(2-pow(p[i].w,2)));
```

```
printf("\n\nn = %i", i);
printf("\n\nw%i = %.16f", i, p[i].w);
printf("\n\npi%i = %.16f", i, p[i].pi);
printf("\n\ndn = %.16f", p[i].dn);
i++;
}
while(p[i-1].w >= p[i-1].w1);
printf("\n\nFin");
printf("\n\nEnter 7 to exit the program\n\n");
scanf("%i", &i);
}
```

Results of the program

The execution of it by the software Turbo C 2.0 produces the following results:

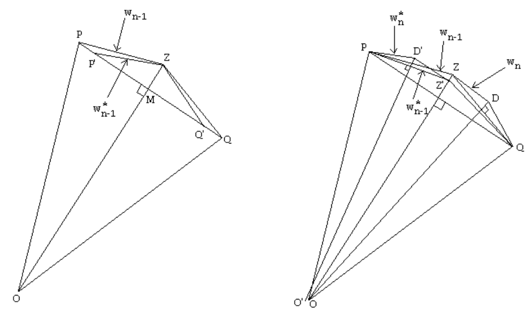
n	$\frac{w_n^*}{R}$	π_n^*
0	0.7653668522834778	3.0614674091339111
1	0.3901805877685547	3.1214447021484375
2	0.1960343122482300	3.1365489959716797
3	0.0981354266405106	3.1403336524963379
4	0.0490827970206738	3.1412990093231201
5	0.0245437510311604	3.1416001319885254
6	0.0122730033472180	3.1418888568878174
7	0.0061376290395856	3.1424660682678223
8	0.0030688035767525	3.1424548625946045

The decrease of π between the seventh and eighth iterations is due to the fact that ignorance accumulated figures of the decimal part w^* from the twenty-third by using the C language from $n=0$ to $n=7$ undervalue this one and therefore that one more and more because of the increasing of 2^{n+2} as calculated π approaches the true π so that $w_7^*/2$ corresponding to $w_7^*/(2R)$ in the program is greater than w_8^* corresponding to w_8^*/R

The program PI.C execution with a replacement condition *while (i <14)*; for the *do* loop products in addition to previous results the following results:

n	$\frac{w_n^*}{R}$	π_n^*
9	0.0015440813731402	3.1622786521511621
10	0.0007720405119471	3.1622779369354248
11	0.0004228639882058	3.4641017913818359
12	0.0002441406250000	4.0000000000000000
13	0.0000000000000000	0.0000000000000000

They then mean that the ignorance of increasingly low numbers combined with 2^{n+2} relatively high for w_8^*/R to w_{12}^*/R completely brings π^* to what is higher than the true π that is to say 4 and that finally the illogical of this treatment results in the invalidity of π^*, w^*/R becoming zero from the thirteenth iteration.



This is justified by the consideration that if $w_{n-1}^* \le w_{n-2}^*/2$ for taking into account the same number of significant digits to calculate w_{n-1}^* as the fact, for example, of taking into account 0.052 for $0.366 \times 0.144 = 0.052704$, then the real w_{n-1} becoming less than w_{n-1}^* that is to say $P'Z = ZQ' < PZ = ZQ$ and w_{n-1}^* and w_{n-2}^* very small, $\Delta w_n^* = w_n^* - w_{n-1}^*/2 \le 0$ that is to say $PZ'/2 > PD'$ with a spacing from the center O of the base circle of radius R to a point O' within a non-circular closed shape regular of center O such that $O'D' = OD$ and $\Delta\pi_n^* = \pi_n^* - \pi_{n-1}^* \le 0$ and vice versa which is why $\pi_8^* < \pi_7^*$ and then as the computer continues to calculate the same number of significant decimal digits very close to true decimals, π_9^* is higher than $\pi_8^*/2$, π_{10}^* than $\pi_9^*/2$ … so that the maximum of π^* is more than just higher than the true π because $w_8^* < w_7^*$ means that more iterations are needed just to reach the ceiling for it.

The maximum error of assessment of w_n, to the inclusion of m significant digits in terms of results below m' thus achieving a maximum error of evaluation of each variable in the program PI.C of 10^{-m}, is Δw_n such that because:

$$\pi_0^* = \frac{2^2 w_0^*}{R} \text{ and } \pi_n^* = \frac{2^{n+2} w_n^*}{R},$$

$$\frac{\pi_n^*}{\pi_0^*} = \frac{2^n w_n^*}{w_0^*} \text{ implies that } w_n^* = \frac{\pi_n^*}{2^n \pi_0^*} w_0^*$$

$$\Delta w_n = w_n - w_n^* = \frac{\pi_n}{2^n \pi_0} w_0 - \frac{\pi_n^*}{2^n \pi_0^*} w_0^* = \frac{\pi_n^*(1+10^{-m})^2}{2^n \pi_0^*(1+10^{-m})} w_0^* - \frac{\pi_n^*}{2^n \pi_0^*} w_0^*$$

$$= \frac{\pi_n^* w_0^*(1+10^{-m}) - \pi_n^* w_0^*}{2^n \pi_0^*} = \frac{\pi_n^* w_0^* 10^{-m}}{2^n \pi_0^*} = \frac{\pi_n^* R 10^{-m}}{2^{n+2}}$$

$$\Delta\left(\frac{w_n}{R}\right) = \frac{\pi_n^* 10^{-m}}{2^{n+2}} = \frac{w_n^*}{R} 10^{-m} < 10^{-m} \text{ car } \frac{w_n^*}{R} < 1$$

which implies that there is no form of palliation to decay to w_8^* because, besides the impossibility of measuring errors of assessment in question accurately, the increment of w_{n-1}^* the maximum error of the assessment before the loop in the program in case it helps to decrease w_n^* despite the inability to increment w_{n-1}^* the error in not having obtained combining the values of variables involved at the same number of significant digits leads to evaluate it, and hence π_n^* and thus the real π which is to be just, on the basis of a polynomial regular with $8 \times 2^7 = 1024$ sides inscribed in the circle with center O, equal to π_7^* and can be re-evaluated over significant digits by increasing possibly the number of iterations for the initial implementation of the program PI.C.

Pi has a maximum in the number of iterations

Algebraically, it is possible to show that π_n admits a maximum in $n \in \mathrm{N}$ by the consideration that:

$$\frac{\partial\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}}{\partial n} = \frac{-\dfrac{\partial\left(\frac{w_{n-1}}{2}\right)^2}{\partial n}}{2\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}} = -\frac{w_{n-1} w_{n-1}'}{4\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}}$$

$$\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2} = \frac{2R^2 - w_n^2}{2R}$$

$$w_{n-1}' = \frac{w_n - w_{n-1}}{n-(n-1)} = w_n - w_{n-1}$$

$$w_n' = \frac{-2R\dfrac{\partial\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}}{\partial n}}{2w_n} = -\frac{\dfrac{\partial\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}}{\partial n}}{w_n} R$$

$$= \frac{w_{n-1} w_{n-1}' R}{4\sqrt{R^2 - \left(\frac{w_{n-1}}{2}\right)^2}\, w_n} = \frac{w_{n-1} w_{n-1}' R}{4 w_n} \frac{2R}{2R^2 - w_n^2}$$

$$= \frac{w_{n-1} w_{n-1}' R^2}{2 w_n (2R^2 - w_n^2)} = \frac{w_{n-1}(w_n - w_{n-1})}{2 w_n \left[2 - \left(\frac{w_n}{R}\right)^2\right]}$$

and the first order condition for this must be such that:

$$\frac{\partial \pi_n}{\partial n} = \frac{1}{R}\left[\frac{2^{n*+3} - 2^{n*+2}}{n^* + 3 - (n^* + 2)} w_{n*} + 2^{n*+2} \frac{\partial w_n}{\partial n}\right] = 0$$

$$\frac{2^{n*+2}}{R}\left[(2-1) w_{n*} + w_{n*}'\right] = 0$$

$$\frac{2^{n*+2}}{R}\left(w_{n*} + w_{n*}'\right) = 0$$

for the maximum number of iterations to calculate π is the solution of the equation insoluble for w_n' known geometrically to be negative $w_{n*} + w_{n*}' = 0$ and it follows that it is impossible to extract the value of n^* from it but a way of monitoring the results obtained automatically on π, is to consider the need that n^* is met when $d_n = 0$ on the basis that it is such that:

$$d_n = \frac{w_n + w_n^{'}}{R} = \frac{1}{R}\left\{ w_n + \frac{w_{n-1}(w_n - w_{n-1})}{2w_n\left[2 - \left(\frac{w_n}{R}\right)^2\right]} \right\}$$

Taking into account, by the execution of the initial program PI.C, the additional training of the *do* loop for d_n yields the following results whose analysis leads to the conclusion that taking into account the rounding errors for calculating the values of the latter to 16 significant digits implies that:

$$d_n = \frac{w_n + w_{n+1} - w_n}{R} = \frac{w_{n+1}}{R}$$

that is to say

$$d_{n-1} = \frac{w_n}{R}$$

This is logical that the maximum for π corresponds to $w_n = 0$ at the end of the $n-1th$ iteration as $w_{n-1} > 0$ and very close to 0

The second-order condition need to be respected by π_n must be such that :

$$\frac{\partial^2 \pi_n}{\partial n^2} = \frac{2^{n^*+3} - 2^{n^*+2}}{R}\left(w_{n^*} + w_{n^*}^{'}\right) + \frac{2^{n^*+2}}{R} w_{n^*}^{''}$$

$$= \frac{2^{n^*+2}}{R}\left(w_{n^*} + w_{n^*}^{'}\right) + \frac{2^{n^*+2}}{R} w_{n^*}^{''} = \pi_{n^*} + \frac{2^{n^*+2}}{R} w_{n^*}^{''}$$

$$= \frac{2^{n^*+2}}{R} w_{n^*}^{''} < -0.732050807 \frac{2^{n^*+2}}{R} w_{n^*+1} < 0$$

Because:

$$w_{n+1}^2 = 2R^2 - 2R\sqrt{R^2 - \left(\frac{w_n}{2}\right)^2}$$

$$R^2 - \left(\frac{w_n}{2}\right)^2 = \left(\frac{2R^2 - w_{n+1}^2}{2R}\right)^2$$

$$\left(\frac{w_n}{2}\right)^2 = R^2 - \left(\frac{2R^2 - w_{n+1}^2}{2R}\right)^2$$

$$w_n = 2\sqrt{R^2 - \left(\frac{2R^2 - w_{n+1}^2}{2R}\right)^2}$$

$$w_n^{''} = \frac{\partial\left(\frac{\partial w_n}{\partial n}\right)}{\partial n} = w_{n+1} - w_n$$

$$= w_{n+1} - 2\sqrt{R^2 - \left(\frac{2R^2 - w_{n+1}^2}{2R}\right)^2}$$

$$= w_{n+1} - 2\sqrt{R^2 - \frac{4R^4 - 4R^2 w_{n+1}^2 + w_{n+1}^4}{4R^2}}$$

n	d_n
1	0.1910327672958374
2	0.0975360199809074
3	0.0490084178745747
4	0.0245344489812851
5	0.0122717078775167
6	0.0061377310194075
7	0.0030704475939274
8	0.0015343781560659

$$= w_{n+1} - 2\sqrt{R^2 - R^2 + w_{n+1}^2 - \frac{w_{n+1}^4}{4R^2}}$$

$$= w_{n+1} - 2\sqrt{w_{n+1}^2 - \frac{w_{n+1}^4}{4R^2}}$$

$$= w_{n+1} - 2w_{n+1}\sqrt{1 - \frac{w_{n+1}^2}{4R^2}}$$

$$= w_{n+1}\left(1 - 2\sqrt{1 - \frac{w_{n+1}^2}{4R^2}}\right)$$

$$\left(2^{n+1}8\right)w_{n+1} = 2\pi R$$

$$\frac{w_{n+1}}{R} = \frac{2\pi}{2^{n+1}8} < \frac{2 \times 3.14...}{8} < 1$$

$$\frac{w_{n+1}^2}{4R^2} < \frac{1}{4}$$

$$1-\frac{w_{n+1}^2}{4R^2} > \frac{3}{4}$$

$$\sqrt{1-\frac{w_{n+1}^2}{4R^2}} > \frac{\sqrt{3}}{2}$$

$$1-2\sqrt{1-\frac{w_{n+1}^2}{4R^2}} < 1-2\frac{\sqrt{3}}{2} = -0.732050807 < 0$$

And I could get the same result by using w_n instead of w_{n+1}

Another method of achieving $w''_{n*} < 0$ is that ensuring that $w''_n < 0$ as :

$$w_{n-1} = 2\sqrt{R^2 - \left(\frac{2R^2 - w_n^2}{2R}\right)^2}$$

$$w_{n-1}^2 = 4\left[R^2 - \left(\frac{2R^2 - w_n^2}{2R}\right)^2\right] = 4\left(R^2 - \frac{4R^4 - 4R^2 w_n^2 + w_n^4}{4R^2}\right)$$

$$= \frac{4R^2 w_n^2 - w_n^4}{R^2}$$

$$w_{n-2}^2 = \frac{4R^2 w_{n-1}^2 - w_{n-1}^4}{R^2}$$

$$\frac{w_n^2}{w_{n-1}^2} = \frac{w_n^2 R^2}{4R^2 w_n^2 - w_n^4} = \frac{R^2}{4R^2 - w_n^2}$$

$$\frac{w_{n-1}^2}{w_{n-2}^2} = \frac{w_{n-1}^2 R^2}{4R^2 w_{n-1}^2 - w_{n-1}^4} = \frac{R^2}{4R^2 - w_{n-1}^2}$$

$\frac{w_n^2}{w_{n-1}^2} < \frac{w_{n-1}^2}{w_{n-2}^2}$ because $w_n < w_{n-1}$ then

$\frac{w_n}{w_{n-1}} < \frac{w_{n-1}}{w_{n-2}}$ and $w''_n < 0$

$\frac{\partial^2 \pi_n}{\partial n^2} < 0$ if $n^* > 1$

$$w_{n-1} = 2\sqrt{R^2 - \left(\frac{2R^2 - w_n^2}{2R}\right)^2}$$

$$w_{n-1}^2 = 4\left[R^2 - \left(\frac{2R^2 - w_n^2}{2R}\right)^2\right] = 4\left(R^2 - \frac{4R^4 - 4R^2 w_n^2 + w_n^4}{4R^2}\right)$$

$$= \frac{4R^2 w_n^2 - w_n^4}{R^2}$$

$$w_{n-2}^2 = \frac{4R^2 w_{n-1}^2 - w_{n-1}^4}{R^2}$$

$$\frac{w_n^2}{w_{n-1}^2} = \frac{w_n^2 R^2}{4R^2 w_n^2 - w_n^4} = \frac{R^2}{4R^2 - w_n^2}$$

$$\frac{w_{n-1}^2}{w_{n-2}^2} = \frac{w_{n-1}^2 R^2}{4R^2 w_{n-1}^2 - w_{n-1}^4} = \frac{R^2}{4R^2 - w_{n-1}^2}$$

$\frac{w_n^2}{w_{n-1}^2} < \frac{w_{n-1}^2}{w_{n-2}^2}$ because $w_n < w_{n-1}$ then

$\frac{w_n}{w_{n-1}} < \frac{w_{n-1}}{w_{n-2}}$ and $w''_n < 0$

$\frac{\partial^2 \pi_n}{\partial n^2} < 0$ if $n^* > 1$

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